Particle Astrophysics and Cosmology (SS 08)
Homework no. 8 (June 18, 2008)

Tutorials: Wednesday, 17:15 to 18:45, AVZ, room 116 (first floor)

1 Virialized systems and rotation curves of galaxies: an introduction to the Dark Matter problem

The virial theorem states that for a closed system of gravitationally interacting point masses the relation $\langle T \rangle = -\langle V \rangle/2$ holds if averaged over time, where T is the kinetic energy and V is the potential energy of the system. Consider, as a typical virialized system, the circular motion of the earth (of mass m) around the sun (of mass M) with radius R and compute the gravitational potential of the earth which is given by

$$\begin{aligned} V(R) &= -\int_{\infty}^{R} \vec{F} \mathrm{d}\vec{r}, \\ \vec{F}(\vec{r}) &= -G_N \frac{M\ m}{r^2}\ \vec{u}_r, \end{aligned} \tag{1}$$

where $\vec{u}_r$ is the unit vector with its origin at the position of the sun, pointing in the direction of $\vec{r}$.

(a) Compute the kinetic energy of the earth. You can find the velocity of the earth by setting equal the gravitational attraction with the centrifugal force. Verify the relation $\langle T \rangle = -\langle V \rangle/2$ which is valid in virialized systems.

The motion of stars in a galaxy can be well described as the motion of particles in a virialized system. In particular the velocity of a star at the distance R from the centre of its galaxy is given by a law $v = v(R)$. The only difference is that here the gravitational force of a spherically symmetric distribution is not generated by a point-like source (as it was for the sun) but by a spherical distribution of density $\rho(r)$ from the centre of the galaxy up to $r = R$. (*Hint*: Remember that according to Newton's theorem the gravitational force due to a spherically symmetric distribution outside of this distribution is the same as the one of a point–like source concentrated in the origin and with a mass equal to the total mass at $r \le R$, whereas the gravitional force from a spherical shell vanishes inside this shell.)

(b) Assuming that the density distribution is given by

$$\rho(r) = \frac{\rho_0}{r^2 (R_0 + r)^{\alpha}}, \tag{2}$$

where ρ_0, R_0 and α are given parameters, derive the law $v = v(R)$ for the stars in the galaxy. Which is the value for α which yields a flat rotational curve ($v =$ *constant*) at $R \gg R_0$?

(c) Start with equation (2) and the value of α which gives a flat rotation curve, given that the Newton constant is

$$G_N = 6.67 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}, \tag{3}$$

compute the total mass of the distribution inside the radius $R = 20$ kpc (recall that $1\mathrm{pc} = 3 \times 10^{16}\mathrm{m}$) which produces a rotation curve with a velocity $\upsilon = 220$ km/s.

2 Velocity distribution of WIMPs

In the galactic rest frame WIMPs are assumed to have the following three–dimensional velocity distribution function:

$$f(\vec{\upsilon})d^3\upsilon = \frac{1}{\pi^{3/2}\upsilon_0^3} e^{-\frac{\vec{\upsilon}^2}{\upsilon_0^2}} d^3\upsilon. \tag{4}$$

(a) Check that

$$\int f(\vec{\upsilon})d^3\upsilon = 1. \tag{5}$$

(b) Compute the one-dimensional WIMP distribution $f_1(\upsilon_x)$ on Earth, where $\vec{\upsilon}_\chi = \vec{\upsilon} + \vec{\upsilon}_e$, $\vec{\upsilon}_e$ is the velocity with which the Earth moves with respect to the galactic center, and $\upsilon_\chi = |\vec{\upsilon}_\chi|$. *Hint*: Use spherical coordinates, and integrate over the angles.

(c) Evaluate $f_1(\upsilon_x)$ for $\vec{\upsilon}_x \ll \vec{\upsilon}_e,\ \vec{\upsilon}_0$.

(d) Evaluate $\upsilon_e(t)$, using

$$\vec{\upsilon}_e = \vec{\upsilon}_\odot + \vec{\upsilon}_\oplus, \tag{6}$$

where $\vec{\upsilon}_\odot = 220$ km/s is the orbital velocity of the Sun around the center of our galaxy and $\vec{\upsilon}_\oplus$ is the orbital velocity of the Earth around the sun.
(*Hint*: Compute $\vec{\upsilon}_\oplus$ from the Earth's orbit, and use the fact that the angle between the path of the Sun around the galaxy and that of the Earth around the Sun is $60°$.)