Particle Astrophysics and Cosmology (SS 08)
Homework no. 7 (to be handed in by June 11, 2008)

Tutorials: Wednesday, 17:15 to 18:45, AVZ, room 116 (first floor)

1 CMB Anisotropies

In class, we expanded the temperature fluctuation in the basis of spherical harmonics.

$$\Delta T(\hat{n}, \vec{x}_O) \equiv \left. \frac{T(\theta,\phi) - T_0}{T_0} \right|_{\vec{x}_O} = \sum_{l,m} a_{lm}(\vec{x}_O) Y^l_m(\hat{n})\,. \tag{1}$$

Here $\vec{x}_O$ describes the location of the observer, and $\hat{n}$ is a unit vector, described by the azimuthal angle ϕ and the polar angle θ:

$$\hat{n} = (\cos\phi \sin\theta,\ \sin\phi \sin\theta,\ \cos\theta)\,; \tag{2}$$

this direction is defined from the location $\vec{x}_O$. The simplest quantity characterizing the anisotropies in the microwave background is the average of a product of two ΔTs,

$$\langle \Delta T(\hat{n}, \vec{x}_O)\, \Delta T(\hat{n}', \vec{x}_O) \rangle,$$

where the averaging is over the observer's location $\vec{x}_O$.

Show that the rotational invariance of the universe requires that the product of two as of Eq.(1) takes the form

$$\langle a_{lm}(\vec{x}_O)\, a_{l'm'}(\vec{x}_O) \rangle = \delta_{ll'} \delta_{m,-m'} C_l \tag{3}$$

from the rotational invariance of the average of a product of two ΔTs, where the averaging is again over $\vec{x}_O$. We can find the coefficients C_l by inverting the Legendre transformation,

$$C_l = \frac{1}{4\pi} \int d^2\hat{n} d^2\hat{n'} P_l(\hat{n} \cdot \hat{n'}) \langle \Delta T(\hat{n}) \Delta T(\hat{n'}) \rangle. \tag{4}$$

Of course, we cannot actually average over positions from which to view the microwave background. What is observed is a quantity averaged over m but not over position:

$$C_l^{obs} \equiv \frac{1}{2l+1} \sum_m a_{lm} a_{l,-m} \tag{5}$$

The fractional difference between the C_l and the observed C_l^{obs} is called *cosmic variance*. Note that it cannot be overcome (without travelling over cosmological distances) and thus poses a fundamental limit to the accuracy with which the power spectrum can be determined!

Fortunately, for Gaussian perturbations, the mean square cosmic variance decreases with l. Show that

$$\left\langle \left(\frac{C_l - C_l^{obs}}{C_l} \right)^2 \right\rangle = \frac{2}{2l+1}. \tag{6}$$

Hint: You can use the identity

$$\begin{aligned} \langle a_{lm} a_{l-m} a_{lm'} a_{l-m'} \rangle &= \langle a_{lm} a_{l-m} \rangle \langle a_{lm'} a_{l-m'} \rangle + \langle a_{lm} a_{lm'} \rangle \langle a_{l-m} a_{l-m'} \rangle \\ &+ \langle a_{lm} a_{l-m'} \rangle \langle a_{l-m} a_{lm'} \rangle \, , \end{aligned} \tag{7}$$

which holds for Gaussian perturbations.

2 Calculation of the WIMP relic density

In class, we saw a simpified estimate of the WIMP relic density. While it correctly reproduces the functional dependence, it gets the numerical coefficient wrong by a few tens of percent, which is much bigger than the current uncertainty on the relic density from observations (within standard ΛCDM cosmology, at least).

Here we will work through a more reliable calculation of the WIMP number density. It is based on the Boltzmann equation,

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle v\sigma_{\rm ann} \rangle \left(n_\chi^2 - n_{\chi,\rm eq}^2 \right) \, . \tag{8}$$

(a) First, re–write the Boltzmann equation in terms of dimensionless quantities, by introducing $Y_\chi \equiv n_\chi / s$, s being the entropy density, and $x = m_\chi / T$. Inserting the explicit expressions for s and H in the radiation–dominated universe, show that this leads to

$$\frac{dY_\chi}{dx} = -\frac{1.32\sqrt{q_{\rm tot}}\, m_\chi M_{\rm Pl}}{x^2} \langle v\sigma_{\rm ann} \rangle \left(Y_\chi^2 - Y_{\chi,\rm eq}^2 \right) \, , \tag{9}$$

where $M_{\rm Pl} = 2.4 \cdot 10^{18}$ GeV is the reduced Planck mass and $q_{\rm tot}$ is the number of relativistic degrees of freedom. Note that we got rid of the $3Hn_\chi$ term on the right–hand side of Eq.(8).

(b) At high temperatures χ should have been (almost) in equilibrium. To treat this phase, write $Y_\chi = Y_{\chi,\rm eq} + \Delta$. For this expansion to make sense, Δ and $\frac{d\Delta}{dx}$ have to be small; one can thus estimate Δ by demading $\frac{d\Delta}{dx} = 0$. In order to determine the "decoupling" temperature, solve the equation

$$\Delta(x_d) = \xi Y_{\chi,\rm eq}(x_d) \, . \tag{10}$$

Show that this gives

$$x_d = \ln \frac{0.382 \xi m_\chi M_{\rm Pl} g_\chi \langle \sigma_{\rm ann} v \rangle}{\sqrt{x_d q_{\rm tot}(x_d)}} \, . \tag{11}$$

Numerically, $\xi = \sqrt{2} - 1$ gives an accurate answer (by comparison with numerical integration of the Boltzmann equation). Eq.(11) can e.g. be solved by iteration.

(c) For $T \leq T_d$, i.e. $x \geq x_d$, the production term in the Boltzmann equation (9) can be ignored. However, χ pair annihilation can still proceed. Under the assumption that $Y_\chi(x \to \infty) \gg Y_\chi(x_d)$, show that this leads to the final answer

$$\Omega_\chi h^2 = \frac{8.7 \cdot 10^{-11} \mathrm{GeV}^{-2}}{\sqrt{q_{\mathrm{tot}}(x_d)} J(x_d)}\,, \tag{12}$$

where the "annihilation integral" is given by

$$J(x_d) = \int_{x_d}^{\infty} \frac{\langle \sigma_{\mathrm{ann}} v \rangle}{x^2} dx\,. \tag{13}$$

Evaluate this integral for the simple case $v\sigma_{\mathrm{ann}} = a + bv^2$, where v is the relative velocity between the annihilating WIMPs. *Hint:* $\langle v^2 \rangle = 6T/m_\chi$.

Note that x_d in this treatment has a different meaning than for the method discussed in class. There, all interactions ceased for $T < T_d$. Here, χ production is also assumed to be negligible for $T < T_d$, but annihilation continues.