Particle Astrophysics and Cosmology (SS 08)
Homework no. 3 (April 30, 2008)

Tutorials: Wednesday, 17:15 to 18:45, AVZ, room 116 (first floor)

1 Horizons

- **Particle Horizon:** Largest distance over which causal connection **can have existed** between any two events. The largest distance from which light signals can have reached us.

- **Event Horizon:** Largest distance over which causal connection **can ever exist** between any two events. This horizon is particular important for black hole physics.

Use these two intuitive definitions of horizons to understand some important properties of the Robertson–Walker space time while comparing it to other spaces. Hint: Consider first the particle horizon by inserting the information on propagation of light into the metrics, respectively. Then determine the integral of the propagated distance:

$$h_p(t) = a(t) \int_{r_i}^{r_0} \mathrm{d}r \,,$$

where subscript i denotes *initial* and 0 denotes *today*. This integral neglects effects of space curvature. Is this a good approximation? Interpret the results.

(a) Determine the particle and event horizon for the Minkowski spacetime.

$$ds^2 = dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \;.$$

(b) Determine the particle and event horizon for the Robertson-Walker spacetime.

$$ds^2 = dt^2 - a^2(t) \left\{dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right\} .$$

(c) Determine the particle and event horizon for the de Sitter spacetime.[1]

$$ds^2 = dt^2 - e^{2Ht} \left\{dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right\} \;.$$

[1]The de Sitter geometry corresponds to the domination of a cosmological constant. The constant expansion rate H and the cosmological constant are related by $H = (\Lambda/3)^{1/2}$.

2 Thermal distribution function of particles

In the lecture the following equations for the number density n, the energy density ρ and the pressure P were given (suppressing time dependence):

$$n = \int \mathrm{d}^3 p\, f(\mathbf{p}) \quad ; \quad \rho = \int \mathrm{d}^3 p\, E f(\mathbf{p}) \quad ; \quad P = \int \mathrm{d}^3 p\, \frac{|\mathbf{p}|^2}{3E} f(\mathbf{p}).$$

The particle momentum $\mathbf{p}$ and its energy E are related by $E^2 = \mathbf{p}^2 + m^2$. The distribution function f is given by

$$f(\mathbf{p})\, \mathrm{d}^3 p = \frac{g}{(2\pi)^3} \frac{\mathrm{d}^3 p}{\exp\left[(E(\mathbf{p}) - \mu)/T\right] \pm 1}$$

By integrating over the distribution function, find the expressions for n, ρ, p in terms of temperature in (a) relativisitic ($T \gg m$), nondegenerate limit ($T \gg \mu$), and (b) nonrelativistic limit ($m \gg T$).

(Hint: You will obtain different formulae for bosons and fermions in (a). The Riemann-zeta function is given as

$$\zeta(s) = \prod_{P(prime)=2}^{\infty} (1 - P^{-s})^{-1},$$

and $\zeta(3) \approx 1.202...$)

In the class, we learned that the entropy density,

$$s \equiv \frac{\rho + P - \sum_i \mu_i n_i}{T} \approx \frac{\rho + P}{T},$$

is *always* dominated by radiation. Does it contradict with the fact that the radiation density is negligible in the current universe? (Recall the value given in Homework no.2)

3 Decoupling of neutrinos from the thermal bath

At high temperatures, neutrinos are kept in thermal equilibrium with the charged leptons by the weak interactions. At a given temperature T_d (to be calculated below) the weak interactions become inefficient and the neutrinos decouple. To have a rough (but very quick!) estimate of T_d one usually compares the interaction rate Γ of the process with the expansion rate H of the universe. Use $H \sim T^2/M_{pl}$; the interaction rate is instead given by $\Gamma \sim v\sigma n$, where $v \sim c = 1$ is the (average) velocity of the neutrinos, and where the (average) cross section σ can be estimated as

$$\sigma \sim G_F^2 E^2, \tag{1}$$

where $G_F \sim 10^{-5}\mathrm{GeV}^{-2}$ is the Fermi constant and E is the energy exchanged in the process. The above relation (2) is valid for energies much smaller than the mass of the $W^\pm$

and Z bosons, $E \ll 80$ GeV. Finally, n is the number density of the neutrinos, $n \sim T^\alpha$. If you do not remember α, you can easily get it by dimensional arguments, remembering that both Γ and T have dimensions of energy (use GeV in this exercise).

(a) Find T_d, by requiring it to be the temperature at which $\Gamma(T_d) = H(T_d)$. Check that $T_d \ll 80$ GeV, so that eq. (2) is indeed valid.

(b) After the neutrinos have decoupled, the thermal bath is made only by $\gamma, e^\pm$. At a temperature $T_\gamma < T_d$, the electron /positrons annihilate, and their energy is transferred to the photons, but not to the neutrinos (since they are decoupled!). As a consequence, the temperature of the photons increases to $T'_\gamma > T_\gamma$, while the ones of the neutrinos remain $T_\nu = T_\gamma$. Calculate T'_γ/T_ν.

(c) In this exercise, you have assumed an instantaneous decoupling of the neutrinos at T_d. Actually, this is not precisely the case, since the decoupling is not a sudden process, but it lasts for some time (approximately, from $T \sim 5$ MeV to $T \sim 0.1$ MeV). When the $e^\pm$ annihilate, the neutrinos are not completely decoupled, and they receive some energy from the annihilation. Precise numerical calculations give $T'_\gamma/T_\nu = 1.399$. Compare it with the analytical result found in the part (2) of this exercise.