Inconsistencies in Special Relativity? Sagnac Effect and Twin Paradox

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And now for something completely different. . .

Special relativity in non-trivial topology: Space and time in rotating systems, Sagnac effect and the twin paradox

Inconsistencies in Special Relativity? Sagnac Effect and Twin Paradox

- Introduction, reminder on special relativity
- Twin paradox
- Rotating frames: Sagnac effect
- Alternative clock synchronization convention
- Local and global effects
- Closed Minkowski space
- Summary

Special Relativity (SR)

Minkowski metric in special relativity in intertial frame F:
proper time $\mathrm{d}s$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \qquad (c = 1)$$

- $\mathrm{d}s^2 > 0$: time-like, e.g. particle velocities $v < c$
- $\mathrm{d}s^2 < 0$: space-like, e.g. directions, separations
- $\mathrm{d}s^2 = 0$: null or light-like, e.g. light propagation

Idea of SR: *Same velocity of light in all frames,*
i.e. coordinate transformation must preserve $\mathrm{d}s^2$

Lorentz Transformation

- Define coordinates for moving frame F': t', x', y', z'
- Assume motion of F' with respect to F with v in x direction
- Assume Minkowski metric in F'

$$
\begin{aligned}
t' &= \gamma(t - vx) & \qquad t &= \gamma(t' + vx') \\
x' &= \gamma(x - vt) & \qquad x &= \gamma(x' + vt') \\
y' &= y & \qquad y &= y' \\
z' &= z & \qquad z &= z'
\end{aligned}
$$

$$\gamma = 1/\sqrt{1 - v^2}$$

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \\
= \quad \mathrm{d}s'^2 &= \mathrm{d}t'^2 - \mathrm{d}x'^2 - \mathrm{d}y'^2 - \mathrm{d}z'^2
\end{aligned}
$$

Time Dilation and Length Contraction

Moving test particle (at rest in F'): x', y', z' constant

$\rightarrow \quad \mathrm{d}t = \gamma \, \mathrm{d}t'$

Observed time is *dilated* by γ

Moving rod measured at certain time t

$\rightarrow \quad \mathrm{d}x = \gamma^{-1} \, \mathrm{d}x'$

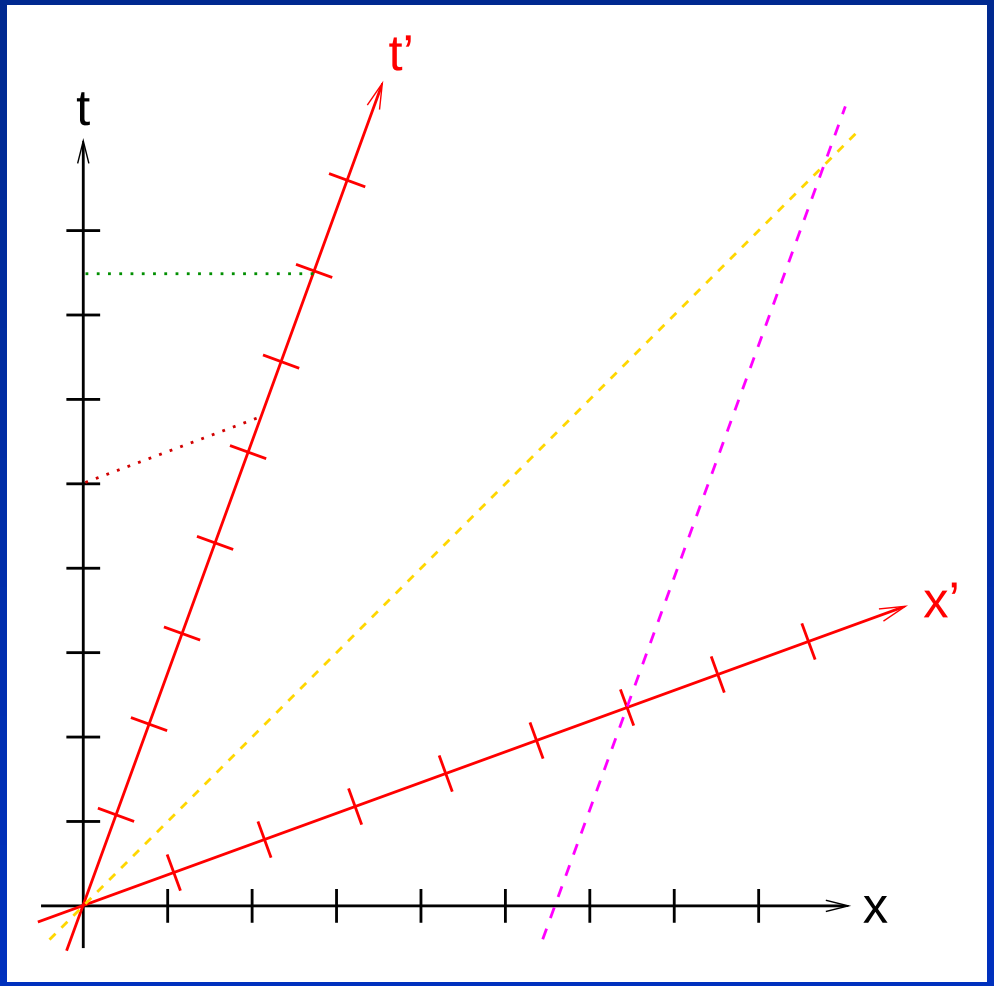
Observed length is *contracted* by γ

Apparent paradoxes:

- Why is t dilated relative to t' but not vice versa?
- Why is x contracted relative to x' but not vice versa?

Depends on exact definitions of lengths and times!

Space Time Diagram



- t' dilated by γ in F
- t dilated by γ in F'
- x' contracted in F
- x contracted in F'

Explanation: Lengths and times are defined as projections!
They do not behave like everyday lengths and times.

Twin Paradox

- One twin F at home with x constant

  Proper time $\Delta s = \Delta t$

- One twin F' travelling with v and back with $-v$

  Proper time $\Delta s' = \Delta t' = \gamma^{-1}\Delta t$

The *travelling* twin ages more slowly!

Paradoxical: Seen from F', F is travelling.
Should then the sister *at home* not be aging more slowly?

Acceleration Effects?

Standard explanation:

- Lorentz transformation connects inertial frames
- Travelling twin has to accelerate to be able to return (in open universe)
- Travelling twin not in inertial frame
- Effect of acceleration or SR not applicable

BUT:

- Age difference is $(1 - 1/\gamma)\Delta t$
- Depends on travel duration and v but not on accelerations!

Calculation in Laboratory Frame

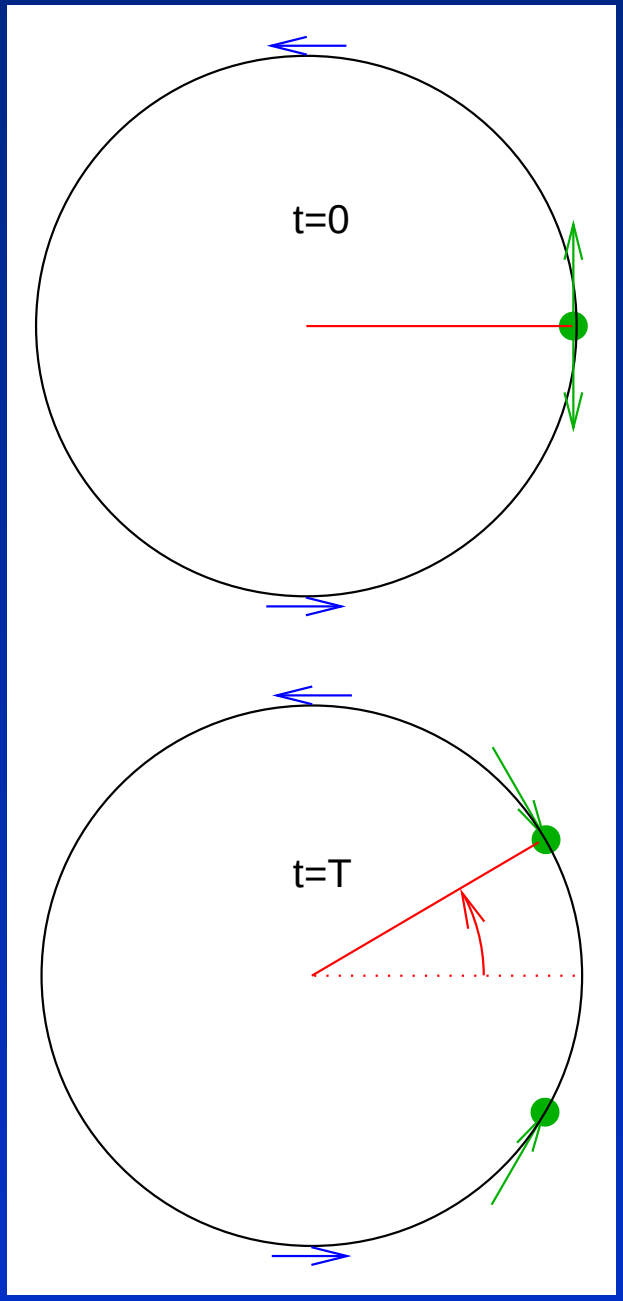
The invariant $\mathrm{d}s$ is the proper time differential.

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 \qquad v(t) = \mathrm{d}x/\mathrm{d}t$$
$$= (1 - v^2)\,\mathrm{d}t^2$$
$$\mathrm{d}s = \gamma^{-1}\,\mathrm{d}t$$

Time dilation is function of v but not of $\dot{v}$!

- *Is it really a physical effect of acceleration?*
  Probably not. . .
- Is F a preferred frame? ('ether frame')
  Yes, but any *global inertial system* is preferred.

Sagnac Effect



- Rotating circle
- Circumference: Length C, velocity v
- Light round trip time on circle depends on direction
  Measurable by interferometry
- Lab frame:
  - ⋆ relative velocity of light: $1 \mp v$
  - ⋆ round trip time: $C/(1 \mp v)$
- On the circle (moving frame):
  - ⋆ velocity of light: $\mathrm{const} \times (1 \mp v)$ **?**
  - ⋆ round trip time: $\mathrm{const}/(1 \mp v)$ **?**

  $\longrightarrow$ Seems to contradict SR

Minkowski Coordinates for Sagnac Effect

- Surrounding Minkowski space: t, ξ, η
  Polar coordinates: t, ϕ, R with $\xi = R\cos\phi$, $\eta = R\sin\phi$

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}t^2 - \mathrm{d}\xi^2 - \mathrm{d}\eta^2 \\ &= \mathrm{d}t^2 - R^2\mathrm{d}\phi^2 - \mathrm{d}R^2 \end{aligned}$$

- Suppress $\mathrm{d}R$ $(R = \mathrm{const})$
- Length on circle $x = R\phi$
- Corotating coordinates: t', x'
  - ⋆ for $x' = 0$: $x = vt$
  - ⋆ $v = R\omega$

Lorentz Transformation for Sagnac Effect

$$x = R\phi \qquad v = R\omega$$

$$x' = R\phi' \qquad \gamma = 1/\sqrt{1-v^2}$$

$$\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \\ \mathrm{d}s'^2 &= \mathrm{d}t'^2 - \mathrm{d}x'^2 \end{aligned}$$

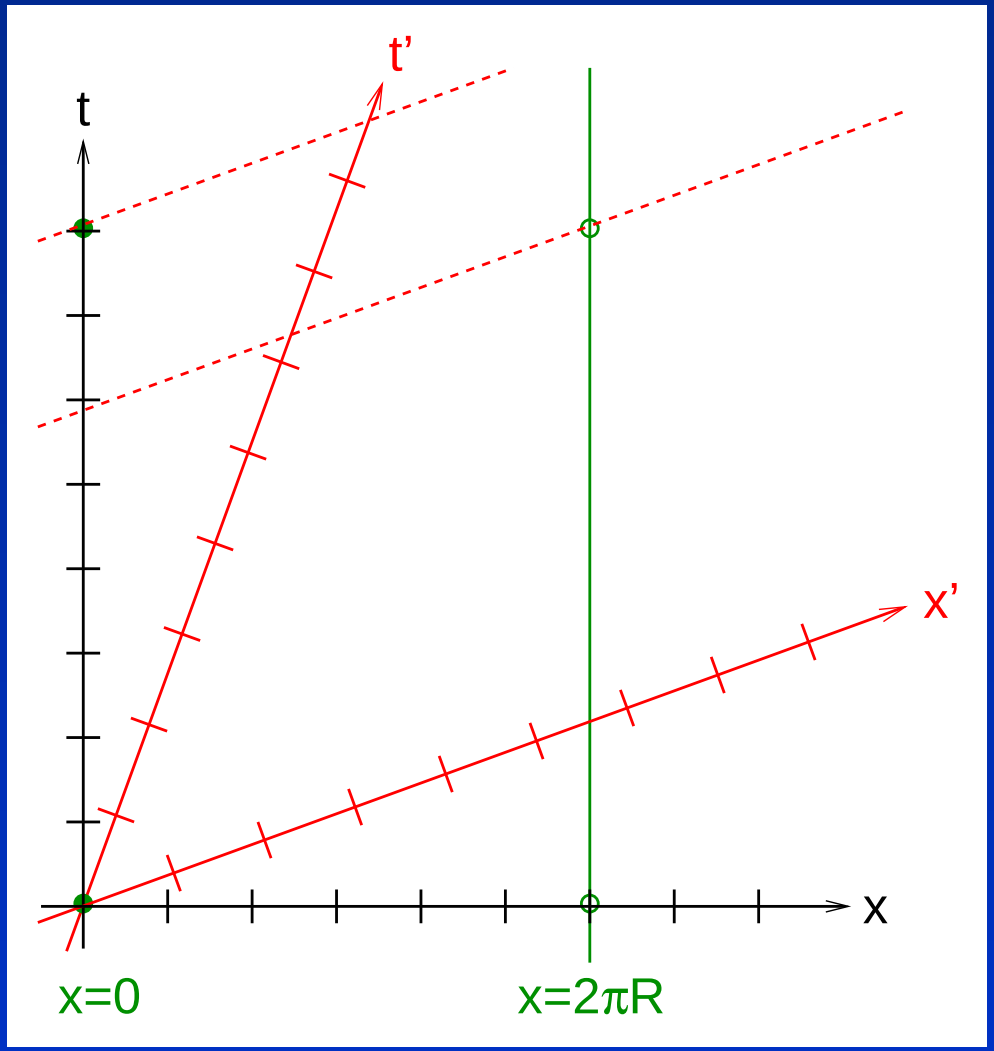
$$\begin{aligned} t' &= \gamma(t - \omega R^2\phi) \\ \phi' &= \gamma(\phi - \omega t) \end{aligned}$$

$$\begin{aligned} t &= \gamma(t' + vx') \\ x &= \gamma(x' + vt') \\ \mathrm{d}s^2 &= \mathrm{d}t^2 - \mathrm{d}x^2 \end{aligned}$$

$$\begin{aligned} t &= \gamma(t' + \omega R^2\phi') \\ \phi &= \gamma(\phi' + \omega t') \end{aligned}$$

Space Time Diagram for Sagnac Effect

If Minkowski metric is valid in rotating frame, how can velocity of light be anisotropic?



- Coordinate x is periodic with $2\pi R$
- For all t identify $x = 0$ with $x = 2\pi R$
- For these identic events
  - ⋆ x' differs
  - ⋆ t' differs!

  $\longrightarrow$ no self-synchroneity

Coordinates x' and t' are not valid *globally*

Generalized Minkowski Coordinates

Metric $\mathrm{d}s^2 = A_{TT}\,\mathrm{d}T^2 - A_{XX}\,\mathrm{d}X^2 - 2A_{XT}\,\mathrm{d}X\mathrm{d}T$

Demands/wishes:

- *T is time* — for $X = \mathrm{const}$: $\mathrm{d}s^2 = \mathrm{d}T^2$ $\rightarrow A_{TT} = 1$
- *X is length* — ($\Leftrightarrow$ Michelson-Morley)
- *Michelson-Morley* — two-way velocity of light is 1 $\rightarrow A_{XT}^2 + A_{XX}A_{TT} = A_{TT}^2$
- *Einstein-Synchronization* — isotropy $\rightarrow A_{XT} = 0$

$\rightarrow$ Minkowski

Alternative Synchronization

- Additional demand: Global coordinates for rotating circle
  Time on circle $T = T(t)$ not a function of x
- Retain *'T is time'*, *'X is length'* and *'Michelson-Morley'* but drop *'Isotropy'*
  $\rightarrow A_{TT} = 1, A_{XT}^2 + A_{XX} = 1$

Leads to $A_{XT} = v$ and

$$T = \gamma^{-1} t \qquad t = \gamma T$$
$$X = \gamma(x - vt) \qquad x = \gamma v T + \gamma^{-1} X$$

and

$$\mathrm{d}s^2 = \mathrm{d}T^2 - (1 - v^2)\,\mathrm{d}X^2 - 2v\,\mathrm{d}X\,\mathrm{d}T$$

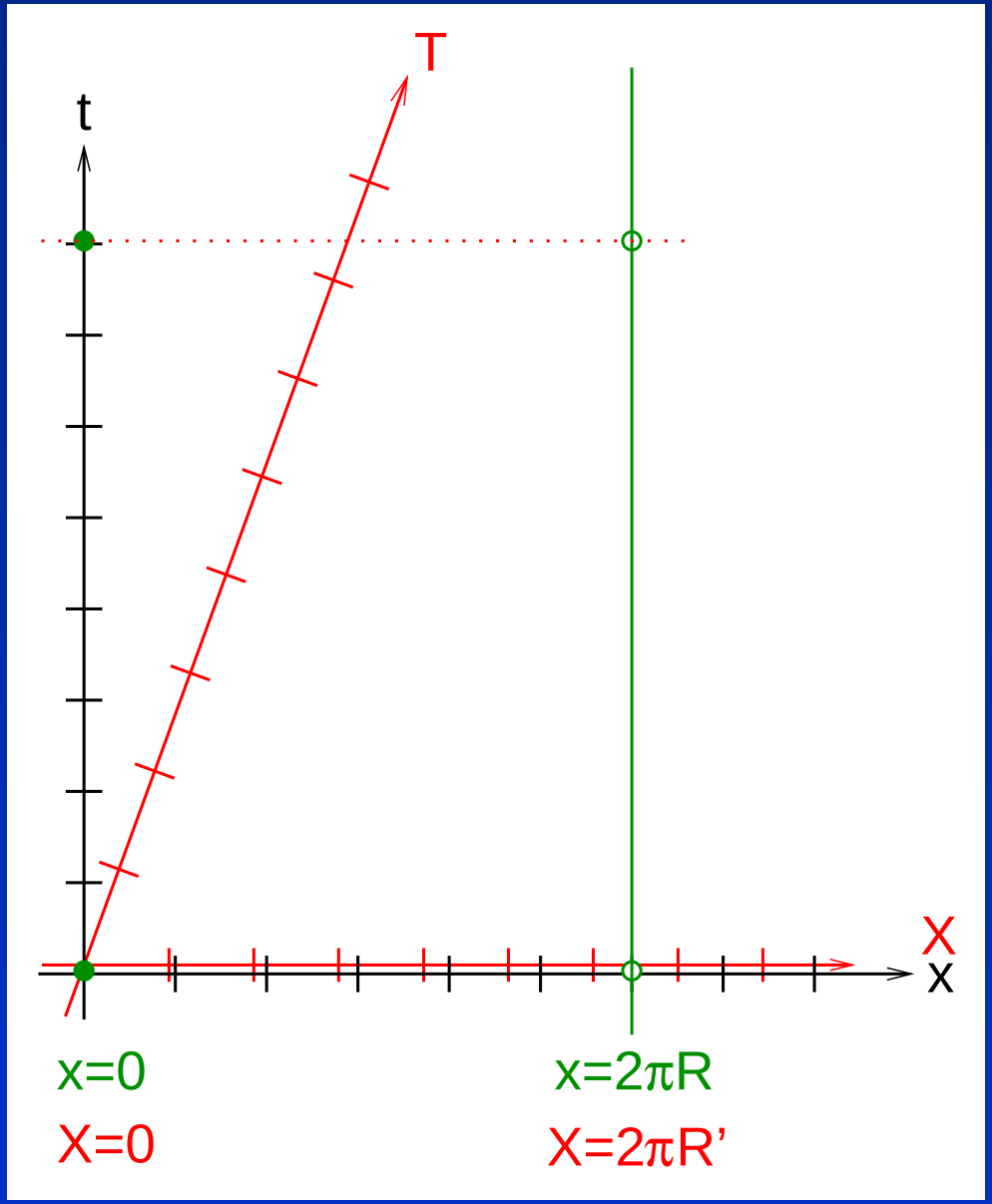
Why 'Synchronization'?

- Time is defined uniquely for certain x modulo constant offset
- Relative offset at x_1 and x_2 is matter of convention
  Synchronization
- Einstein synchronization: isotropy

Standard Lorentz: t', x'
Alternative: T, X

$$X = x'$$
$$T = t' + vx'$$

Alternative Space Time Diagram



- Global coordinates and synchronization
- Each event synchronized with itself
- Periodicity also in X with $2\pi R' = 2\pi R\,\gamma$
- Incompatible with
  - ⋆ Einstein synchronization
  - ⋆ slow clock transport
- Anisotropic velocity of light

Lengths in Alternative Synchronization

Formal definition of physical units:

- Time: *Second* is $9\,192\,631\,770$ vibrations of $^{133}\mathrm{Cs}$ transition
- Velocity of light $c := 299\,792\,458\,\mathrm{m\,s^{-1}}$
- Lengths: $\Delta L = c\,\Delta t \rightarrow$ *Meter*
- One-way velocity of light not uniquely defined
- Use two-way velocity

$\rightarrow$ coordinate X is *length* in rotating system

- Length contraction *and expansion:* $C = C'/\gamma$
  (circumference $C = 2\pi R$, $C' = 2\pi R'$)

Discussion of Sagnac Effect

- Same question as for twin-paradox: *Is it an acceleration effect?*
- Answer: Probably not. No acceleration terms in metric

Twins on rotating circle:

- All twins will meet again without acceleration (relative to circle)
- Twin travelling with $\mathrm{d}X/\mathrm{d}T = -\gamma^2 v$ ages most rapidly
- . . . and is at rest with respect to the outside world
- Is there a preferred frame?
  Yes!
- Two light rays equivalent to two twins

Local SR

Locally on circle and globally in open space time:

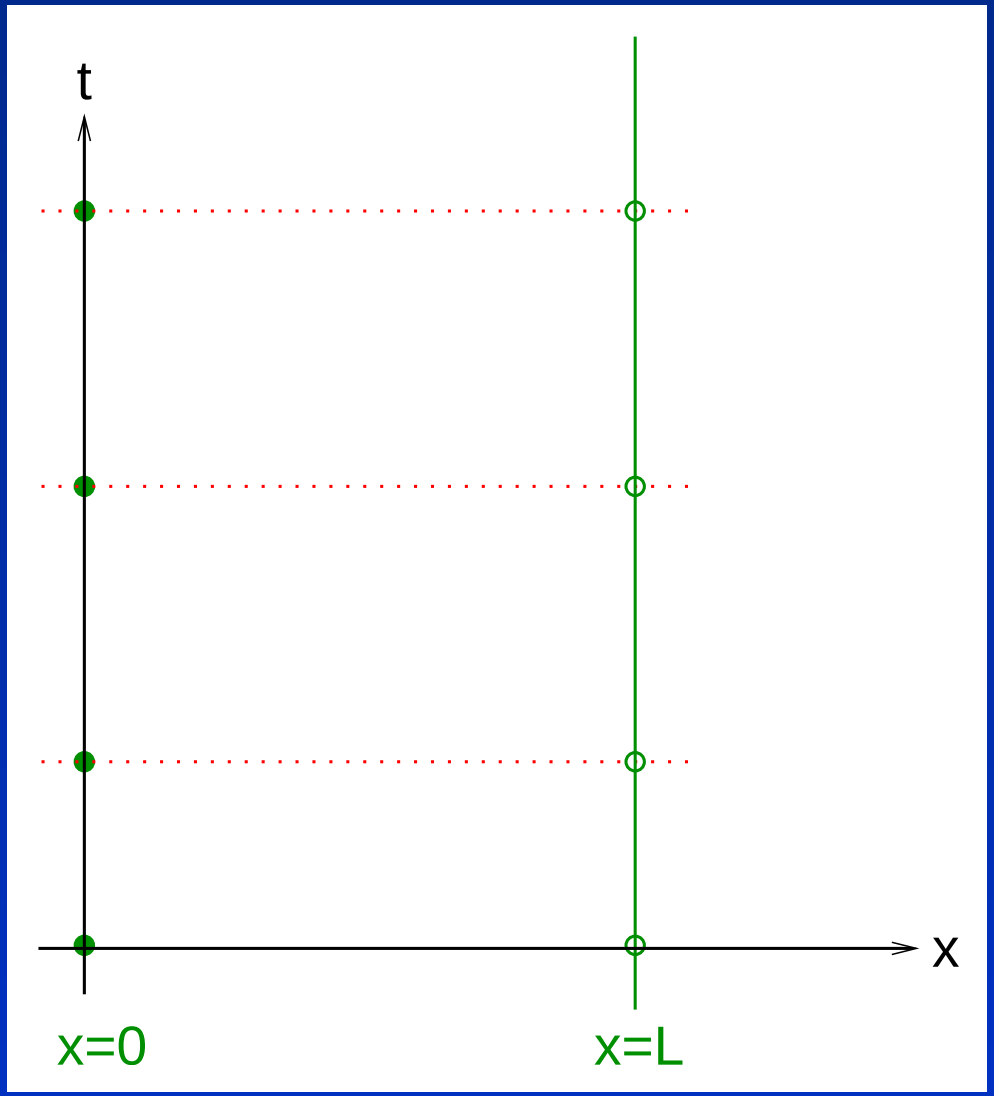
- Minkowski coordinates and Lorentz transformation best choice
- Isotropic velocity of light
- Clock synchronization: Einstein or slow transport
- No preferred inertial frame
- Twin paradox

Global SR

Globally on circle:

- Minkowski coordinates not valid
- Anisotropic velocity of light
- Clock synchronization: Einstein in both directions, take average
- Preferred 'inertial' frame
- Twins not paradoxical

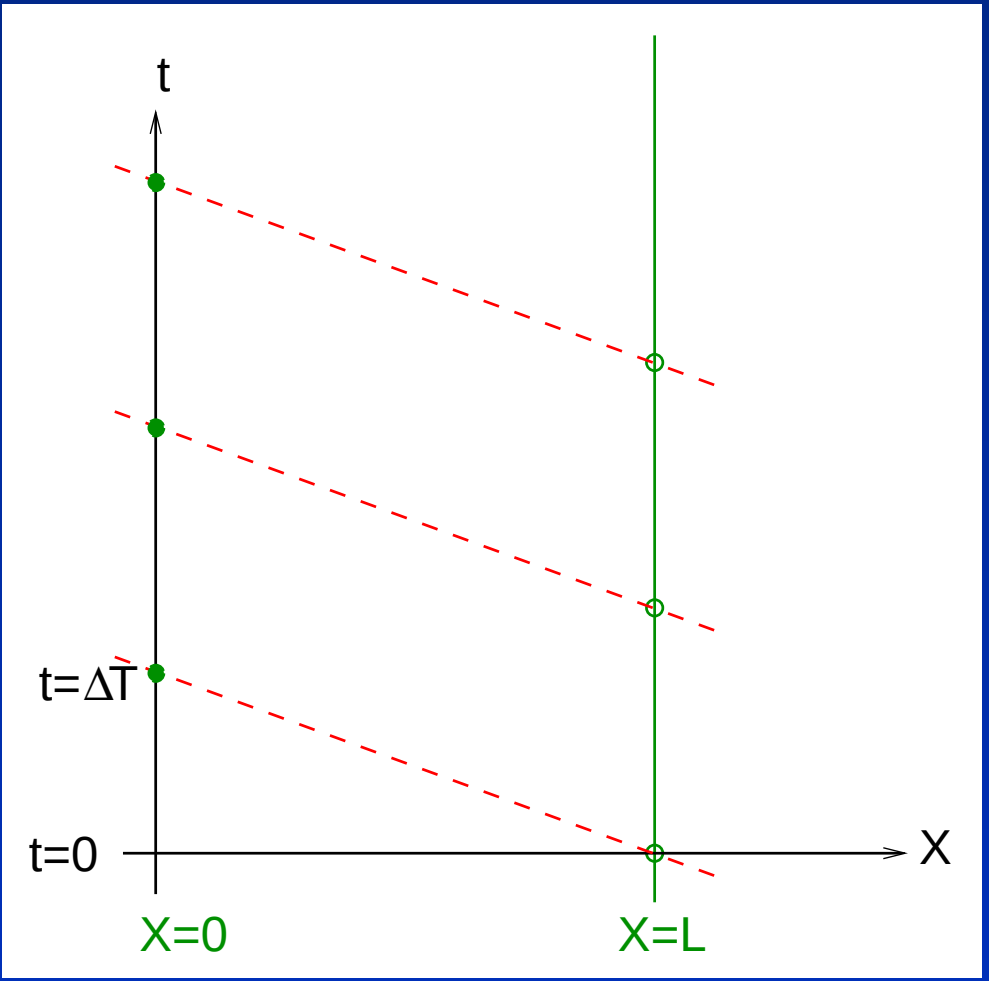
It's the Topology, Stupid



SR fails only if going around completely: *Try closed Minkowski space time!*

- Start with flat Minkowski space time (1+1 dim)
- Clip at $x = 0$ and $x = L$
- Glue together pieces with equal t
- Result equivalent to circle at rest
- Moving inertial frames not all equal

Let Minkowski Rotate



- Start with Minkowski
- Same cuts
- Glue together with time offset $\Delta T < L$
- Resynchronize to remove discontinuity
  $T = t + vx,$
  $v = \Delta T / L$

$$\mathrm{d}s^2 = \mathrm{d}T^2 - (1 - v^2)\,\mathrm{d}X^2 - 2v\,\mathrm{d}X\,\mathrm{d}T$$

same as
rotating circle !

The Preferred Inertial Frame

- Length of 'rotating' Minkowski frame: L
- Light travel times: $L(1 \pm v) = L\gamma^{-2}/(1 \mp v)$

New frame: t, x moving relative to T, X with $\mathrm{d}X/\mathrm{d}T = -\gamma^2 v$
Transformation as before:

$$T = \gamma^{-1} t \qquad t = \gamma T$$
$$X = \gamma(x - vt) \qquad x = \gamma v T + \gamma^{-1} X$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$$

- Length of special frame: $\gamma^{-1}\, L$ (minimal)
- Light travel times: $\gamma^{-1}\, L$ (equal)

Summary (1)

- SR conventions fine locally
  - ⋆ Einstein synchronization, slow clock transport
  - ⋆ Isotropic velocity of light
- Strange effects for closed curves
  - ⋆ Twin paradox
  - ⋆ Paradoxical Sagnac effect
- Global effects different from local ones
- Blame acceleration?
- Alternative synchronization
  - ⋆ No paradoxes but preferred inertial frame

Summary (2)

- Minkowski space time with closed topology
  - ⋆ Same effects as on rotating circle
  - ⋆ SR conventions not valid globally
  - ⋆ Clock synchronization
  - ⋆ Anisotropy of light propagation
  - ⋆ Preferred inertial frames
  - ⋆ *No acceleration effects!*

Conclusion: SR conventions only good locally/for open topology

What does this mean?

Loose Ends

- Physics in closed topology
- What about closed *time-like* curves?
- Lengths and rulers
- Geometry of rotating disc: Ehrenfest Paradox
- Global effects can be described as local ones (e.g. least action as differential equations)
- Is local description of physics complete?

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