Particle Astrophysics and Cosmology (SS 08)

Homework no. 2 (April 23, 2008)

Tutorials: Wednesday, 17:15 to 18:45, AVZ, room 116 (first floor)

1 Living on a 3-sphere

It was shown in the lecture that a line-element $(d\vec{x})^2$ on the surface of a 2 sphere with radius R_c is

$$(d\vec{x})^2 = \frac{(dr)^2}{1 - \frac{r^2}{R_c^2}} + r^2 (d\theta)^2 \tag{1}$$

This already looked somewhat similar to the spatial part of the line-element of the Robertson-Walker metric, given by

$$(ds)^2 = (dt)^2 - a^2(t) \left[\frac{(dr)^2}{1 - \frac{kr^2}{R_c^2}} + r^2((d\theta)^2 + \sin^2\theta (d\psi)^2) \right] \tag{2}$$

Calculate $(d\vec{x})^2$ for a 3-sphere embedded in a flat 4-dimensional space, and compare with the Robertson-Walker metric. *Hint:* Use spherical coordinates for the 3-dimensional surface of the sphere and $x_1^2 + x_2^2 + x_3^2 + x_4^2 = R_c^2$, where x_4 is the fictional 4th spatial dimension.

2 The $\Omega_{\Lambda,0} - \Omega_{m,0}$ plane

In this exercise we will discuss the behaviour of the universe depending on the parameters $\rho_{\Lambda,0}$ and $\rho_{m,0}$, i.e. the dominant contributions to today's energy density. The radiation will be neglected in most of the following arguments. (Recall: $\rho_{r,0} \sim 10^{-4}\rho_{\text{crit},0}, \rho_{m,0} \sim 0.25\rho_{\text{crit},0}$ and $\rho_{\Lambda,0} \sim 0.75\rho_{\text{crit},0}$, where $\rho_{\text{crit},0}$ is today's value of the critical energy density, corresponding to a flat Universe.)

2.1 Friedmann equation and the Cosmic sum rule

In class the first Friedmann equation was written in the form

$$H^2 = \frac{\rho}{3M_P^2} - \frac{k}{R_c^2 a^2}\,, \tag{3}$$

where $H = \dot{a}/a$ is the Hubble parameter, ρ is the energy density, M_P is the reduced Planck mass, and R_c is the curvature radius of the Universe. In order to derive the "cosmic sum rule", divide both sides of this equation by H^2, split ρ into the sum of matter and radiation densities and a contribution from a cosmological constant, and express these densities in terms of their current values and appropriate powers of a. *Hint:* The result looks a bit neater if expressed in terms of the critical energy density. The ratios of actual densities ρ_i and the critical density are usually denoted by Ω_i.

2.2 Flat, static and constantly expanding universe

Set $\Omega_{r,0} = 0$.

(a) Derive the relation between $\Omega_{\Lambda,0}$ and $\Omega_{m,0}$ for a flat universe.

(b) Derive the relation between $\Omega_{\Lambda,0}$ and $\Omega_{m,0}$ for a universe with constant expansion rate today, $\ddot{a} = 0$. *Hint:* Use the second Friedmann equation.

(c) Derive the relation between $\Omega_{\Lambda,0}$ and $\Omega_{m,0}$ for a static universe, $\dot{a} = 0$, as first envisioned by Einstein. Is this solution stable against small perturbations?

2.3 The (dark?) fate of the universe

Again set $\Omega_{r,0} = 0$.

(a) Consider an empty universe, $\Omega_{m,0} = 0$. Use the first Friedmann equation and the cosmic sum rule to discuss the early and late time evolution of the universe.

(b) Consider a pure matter universe, $\Omega_{\Lambda,0} = 0$. Use again the first Friedmann equation and the cosmic sum rule to discuss the early and late time evolution of the universe.

(c) Indicate the border line between eternally expanding and recollapsing universes in the $\Omega_{\Lambda,0}$ vs. $\Omega_{m,0}$ plane.