Particle Astrophysics and Cosmology (SS 08)

Homework no. 10 (July 2, 2008)

Tutorials: Wednesday, 17:15 to 18:45, AVZ, room 116 (first floor)

1 Scalar Field Dynamics

For constructing dynamical models both of Dark Energy and of cosmological inflation, the basic ingredient is a scalar field. In this exercise, we will derive the equation of motion of it. The result will be used in the following exercise.

Consider the action for a minimally coupled scalar field ϕ,

$$S = \int d^4x\sqrt{g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] . \tag{1}$$

For flat FRW metric, the physical spatial volume element satisfies

$$d^3x\sqrt{g} = a^3 d^3\tilde{x} , \tag{2}$$

where a is the scale factor and $\tilde{x}_i$ are co–moving spatial coordinates (i.e. $d^3\tilde{x}$ is independent of time). Show that a spatially homogeneous field satisfies the equation of motion

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + \frac{dV(\phi)}{d\phi} = 0,$$

where $H = 1/a\ (da/dt)$ as usual. Convince yourself that in the appropriate limit this reproduces the Klein–Gordon equation, $\Box\phi + m^2\phi = 0$, if $V(\phi) = \frac{1}{2}m^2\phi^2$. *Hint:* Use the stationarity of the action of Eq.(1); since ϕ has no dependence on spatial coordinates, you can ignore the $\int d^3\tilde{x}$ part of the integral.

2 A Possible Candidate of Dark Energy - Quintessence?

In exercise 6, we have seen that the universe is undergoing accelerating expansion, and that it implies that the vacuum energy is only a few times greater than the present matter density. In this exercise, we will study a time–varying vacuum energy which is sometimes called *quintessence*.

For simplicity, let us consider a single real scalar field $\phi(\vec{x}, t)$. As we derived above, the equation of motion for a homogeneous field in a time–dependent FRW background metric is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{3}$$

where H is the (time-dependent) Hubble parameter. Consider a potential $V(\phi)$ of the form

$$V(\phi) = M^{4+\kappa}\phi^{-\kappa}, \tag{4}$$

with an arbitrary κ (> 0).

(a) By solving Eq.(3) with the potential (4), show that at very early times ρ_ϕ must have been less than ρ_R. *Hint:* One can assume radiation dominance at sufficiently early times. You will find a solution

$$\phi = \left(\frac{\kappa(2+\kappa)^2 M^{4+\kappa} t^2}{6+\kappa} \right)^{\frac{1}{2+\kappa}} . \tag{5}$$

Note that this solution is not unique.

(b) Show that any other solution that comes close to the one in (a) will approach it as t increases. Therefore, it is called a *tracker solution.*

(c) What happens as time elapses? Find ϕ at sufficiently late times, i. e. in the ρ_ϕ dominated era. *Hint:* The "damping" term in Eq.(3) will eventually slow the growth of ϕ, therefore $\dot{\phi}^2$ will become less than $V(\phi)$. Also, $\ddot{\phi}$ will become negligible.