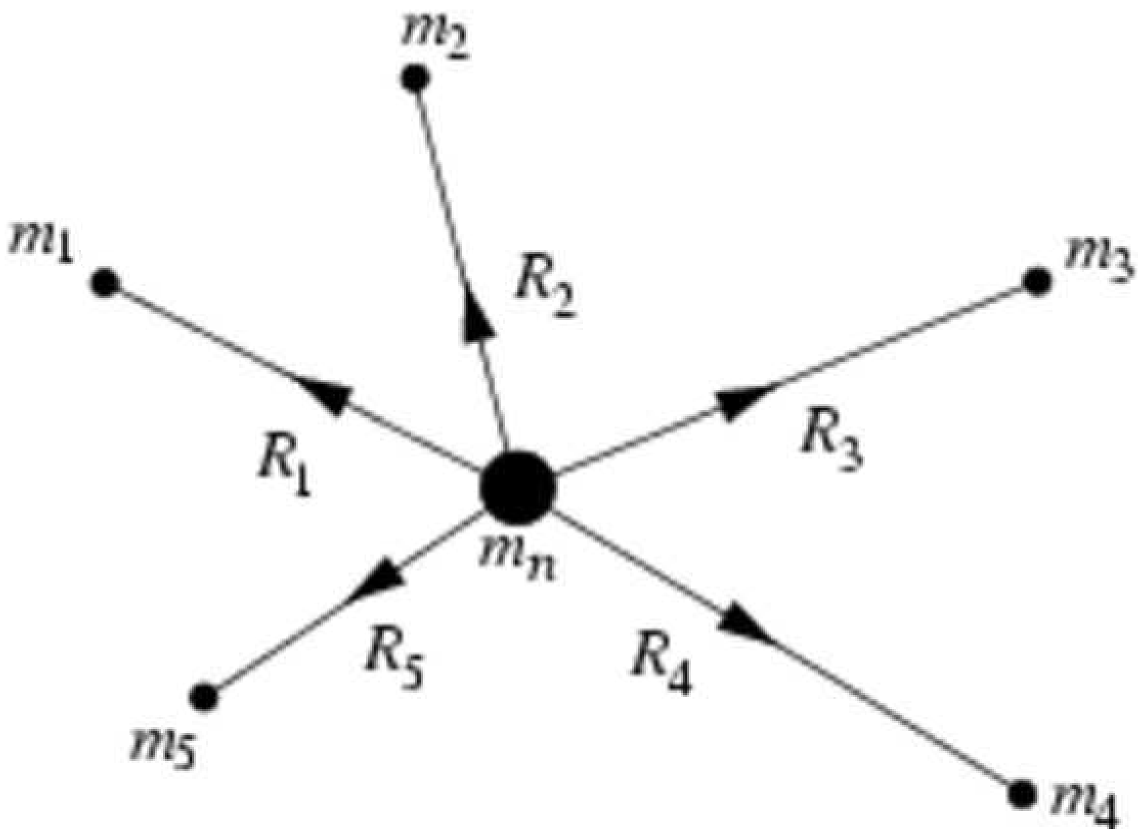
Post-Newtonian Simulations

Sverre Aarseth

Institute of Astronomy, Cambridge

- Introduction
- Wheel-spoke regularization
- Initial conditions
- Post-Newtonian treatment
- Idealized systems
- White dwarf models
- Conclusions



m_1
m_2
m_3
m_4
m_5
m_n
R_1
R_2
R_3
R_4
R_5

Useful equations

Wheel-spoke Hamiltonian

$$H = \sum_{i=1}^{n} \frac{\mathbf{p}_i^2}{2\mu_i} + \frac{1}{m_0} \sum_{i<j}^{n} \mathbf{p}_i^{\mathrm{T}} \cdot \mathbf{p}_j - m_0 \sum_{i=1}^{n} \frac{m_i}{R_i} - \sum_{i<j}^{n} \frac{m_i m_j}{R_{ij}}$$

Inverse transformations

$$\mathbf{q}_i = \frac{1}{2} \mathbf{A}_i^{\mathrm{T}} \mathbf{Q}_i \,, \qquad \mathbf{p}_i = \frac{1}{4} \frac{\mathbf{A}_i^{\mathrm{T}} \mathbf{P}_i}{R_i}$$

Coalescence condition

$$r_{\mathrm{coal}} = \frac{6(m_i + m_0)}{c^2} \,, \qquad c = \frac{3 \times 10^5}{V^*}$$

GR radiation time-scale

$$\tau_{\mathrm{GR}} = \frac{5 a^4 c^5}{64 m_i m_0^2} \frac{(1 - e^2)^{7/2}}{g(e)} \,,$$

Kozai cycles $\qquad \cos^2 i \,(1 - e_{\mathrm{in}}^2) = \mathrm{const}$

Time-scale for Kozai cycle

$$T_{\mathrm{Kozai}} = \frac{T_{\mathrm{out}}^2}{T_{\mathrm{in}}} \left(\frac{1 + q_{\mathrm{out}}}{q_{\mathrm{out}}} \right) (1 - e_{\mathrm{out}}^2)^{3/2} f(e_{\mathrm{in}}, \omega_{\mathrm{in}}, \psi)$$

GR capture radius

$$r_{\mathrm{cap}} = b \left[\frac{m_0 m_i (m_0 + m_i)^{3/2}}{c^5 v_\infty^2} \right]^{2/7}$$

Initial conditions

Density distribution

$$\rho(r) \;=\; \frac{1}{r^{1/2}(1+r^{5/2})}$$

Mass function

$$N = 10^5, \quad m_0 = 3.0 \times 10^{-3}, \quad m_i = 1.0 \times 10^{-5}$$

Velocity dispersion

$$\sigma^2(r) = \frac{1}{\rho(r)} \int_r^\infty \frac{\rho(r)}{r^2}[m(r) + m_0]dr$$

Half-mass radius

$$r_{\rm h} \simeq 0.09 {\rm pc}$$

White dwarf models

$$r_{\rm h} \simeq 0.9 \text{ or } 2.7 \text{ pc}\,, \quad r^* = 5 \times 10^{-5} \text{ AU}$$

Disruption radius

$$r_t = \left(\frac{m_0}{m_i}\right)^{1/3} r^*$$

Post-Newtonian Terms

Equation of motion $$\frac{d^2\mathbf{r}}{dt^2} = \frac{M}{r^2}\left[(-1+A)\frac{\mathbf{r}}{r} + B\mathbf{v}\right]$$

First-order precession $$M = m_1 + m_2, \quad \eta = \frac{m_1 m_2}{M^2}$$

$$A_1 = 2(2+\eta)\frac{M}{r} - (1+3\eta)v^2 + \frac{3}{2}\eta\dot{r}^2$$

$$B_1 = 2(2-\eta)\dot{r}$$

Higher-order precession $$A_2 = ..., \quad B_2 = ..., \quad A_3 = ..., \quad B_3 = ...$$

Gravitational radiation $$A_{5/2} = \frac{8}{5}\eta\frac{M}{r}\dot{r}\left(\frac{17M}{3r} + 3v^2\right)$$

$$B_{5/2} = -\frac{8}{5}\eta\frac{M}{r}\left(3\frac{M}{r} + v^2\right)$$

Total GR perturbation

$$\mathbf{P}_{GR} = \frac{m_1 m_2}{c^2 r^2}\left[(A_1 + \frac{A_2}{c^2} + \frac{A_{5/2}}{c^3})\frac{\mathbf{r}}{r} + (B_1 + \frac{B_2}{c^2} + \frac{B_{5/2}}{c^3})\mathbf{v}\right]$$

Radiation energy loss $$\Delta E_{GR} = \int \mathbf{P}_{GR} \cdot \mathbf{v}\, dt$$

Periapse advance $$\Delta\omega = \frac{6\pi m_0}{c^2 a(1-e^2)}$$

Decision-making

Softening of singular terms $\quad \epsilon = \epsilon_0 R_{\rm grav}\,, \quad \epsilon_0 = 10^{-3}$

Perturber selection $\quad d < \left[\frac{2m_j}{m_0\gamma_0}\right]^{1/3} R_{\rm grav}\,, \quad \gamma_0 = 10^{-6}$

Subsystem membership $\quad d < 2R_{\rm cl}\,, \quad \dot{d} < 0$

Progressive time-scales

2.5PN for $\tau_{\rm GR} < 1000$

1PN for $\tau_{\rm GR} < 100$

2PN for $\tau_{\rm GR} < 50$

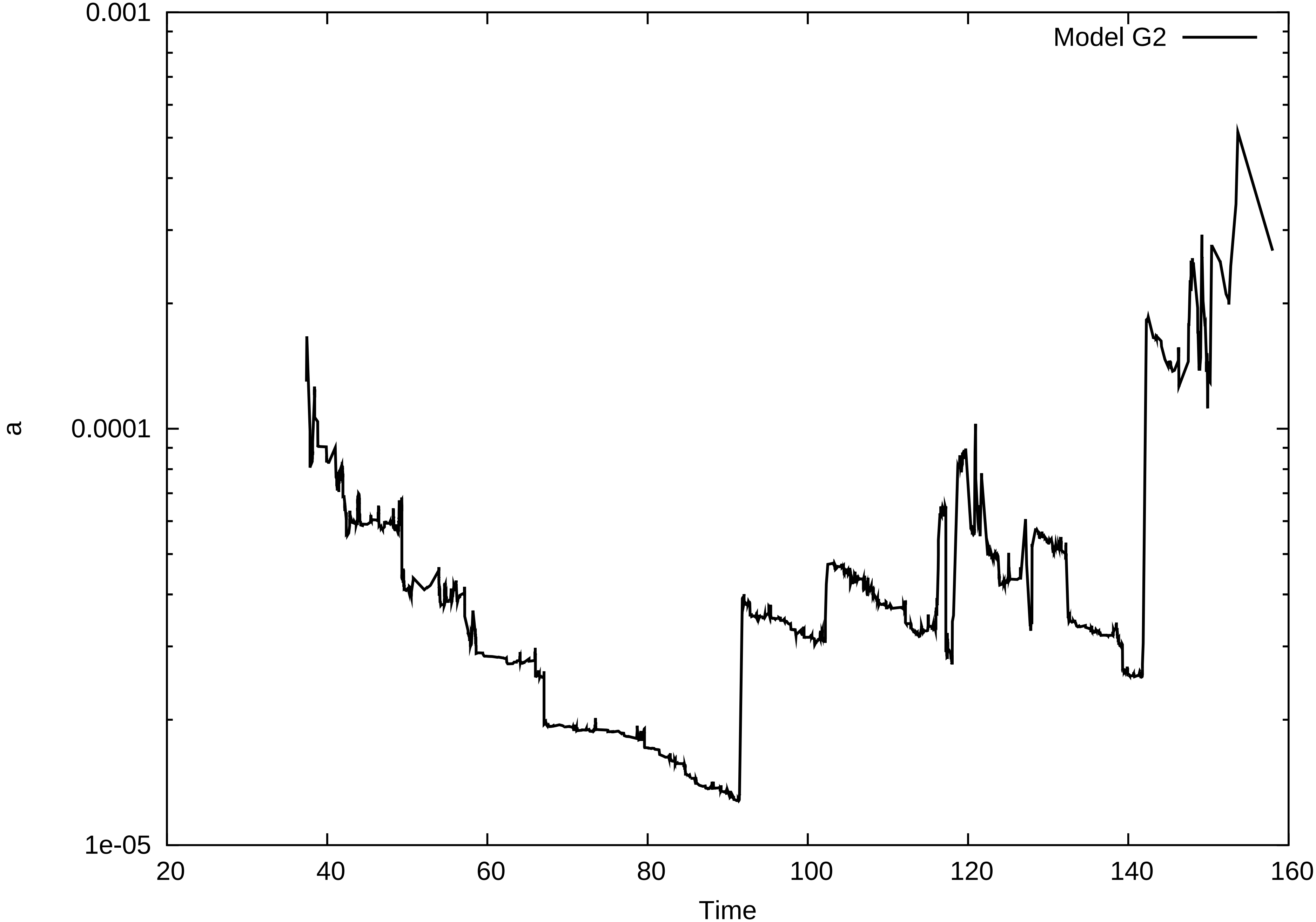
3PN for $\tau_{\rm GR} < 10$

Kozai time-scale $\quad T_{\rm Kozai} < 10$

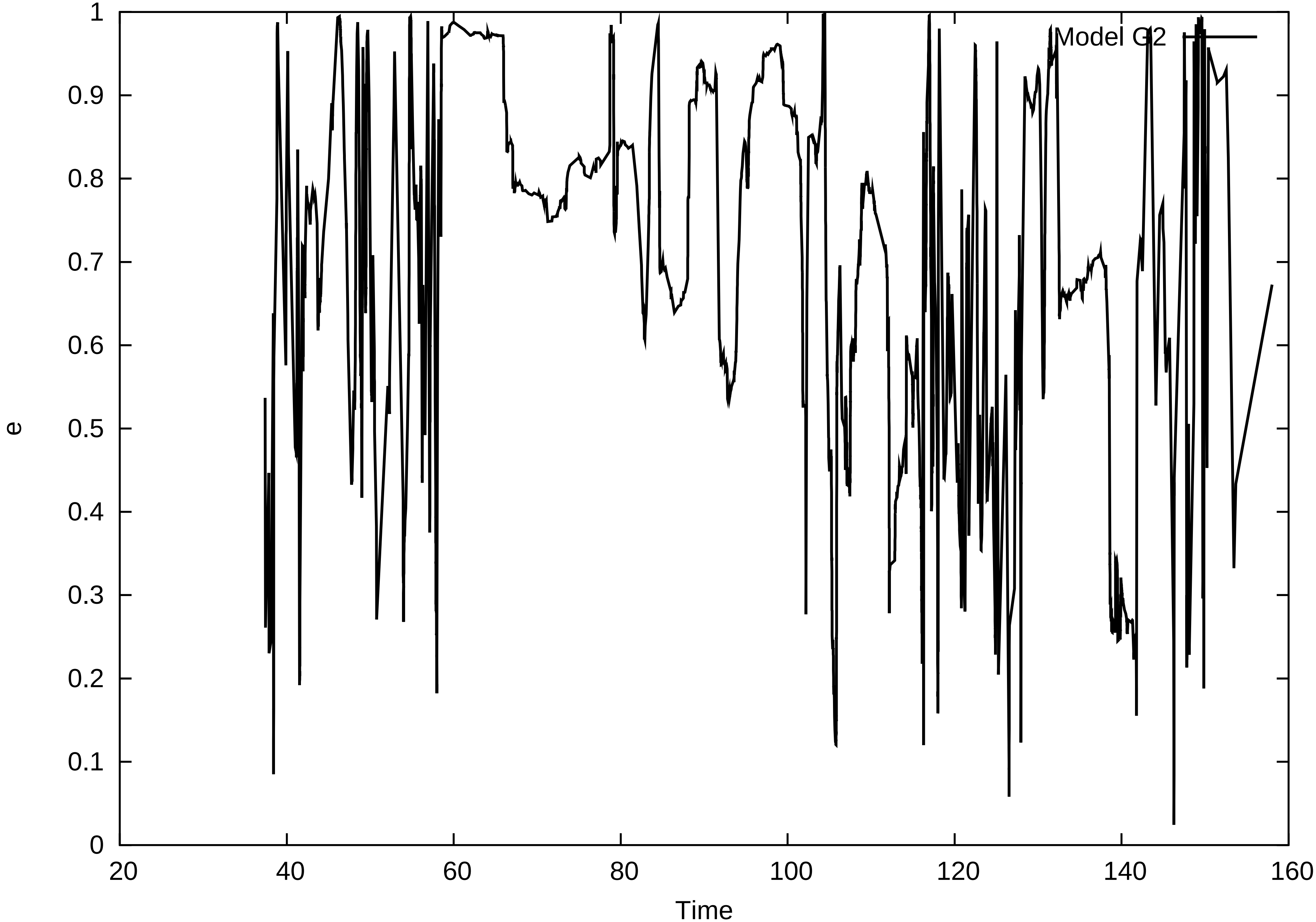
Profiling with GRAPE-6

Wheel-spoke: 6.5 % of CPU, $\quad N = 10^5$

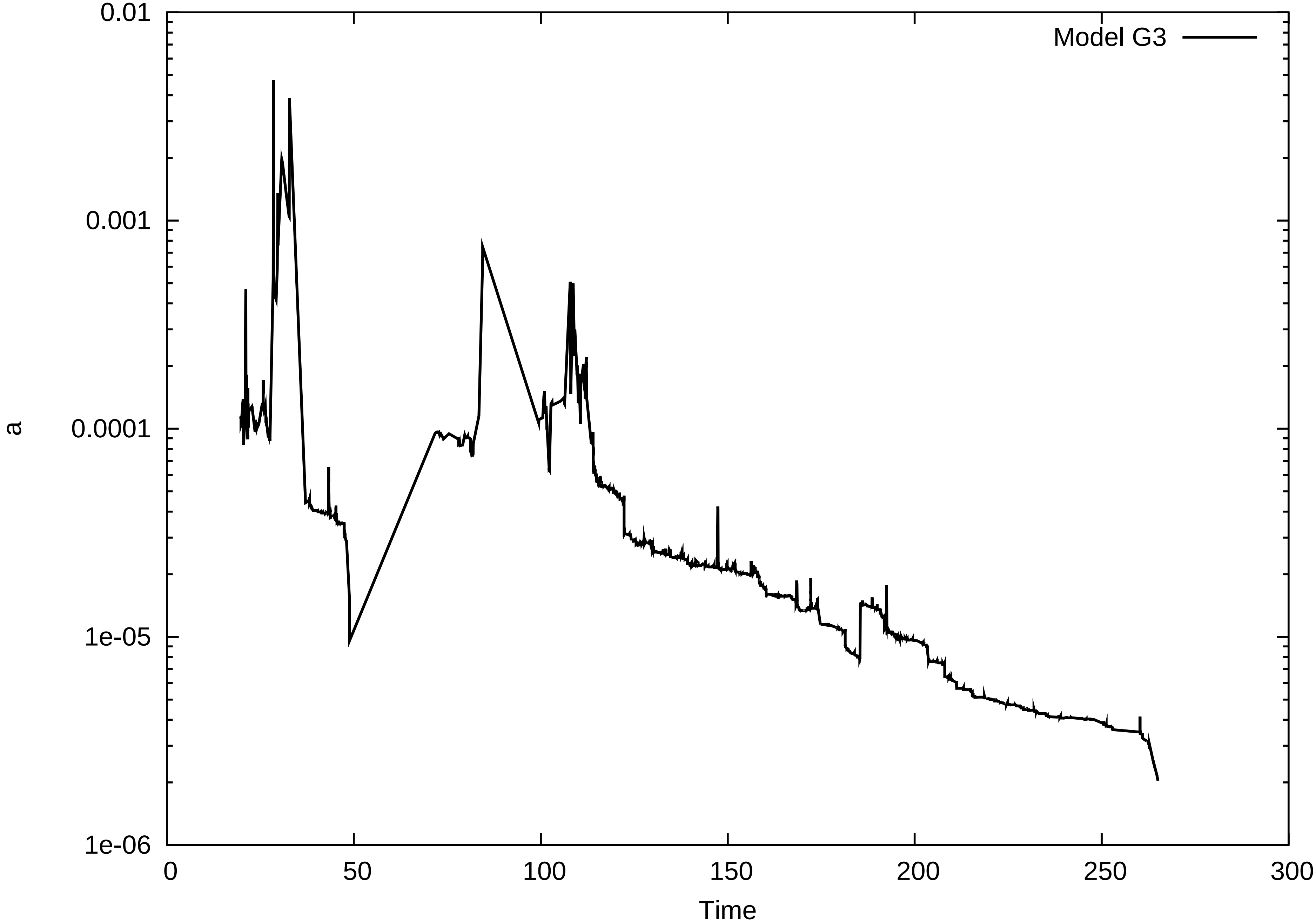
N-body part: 14 % of CPU



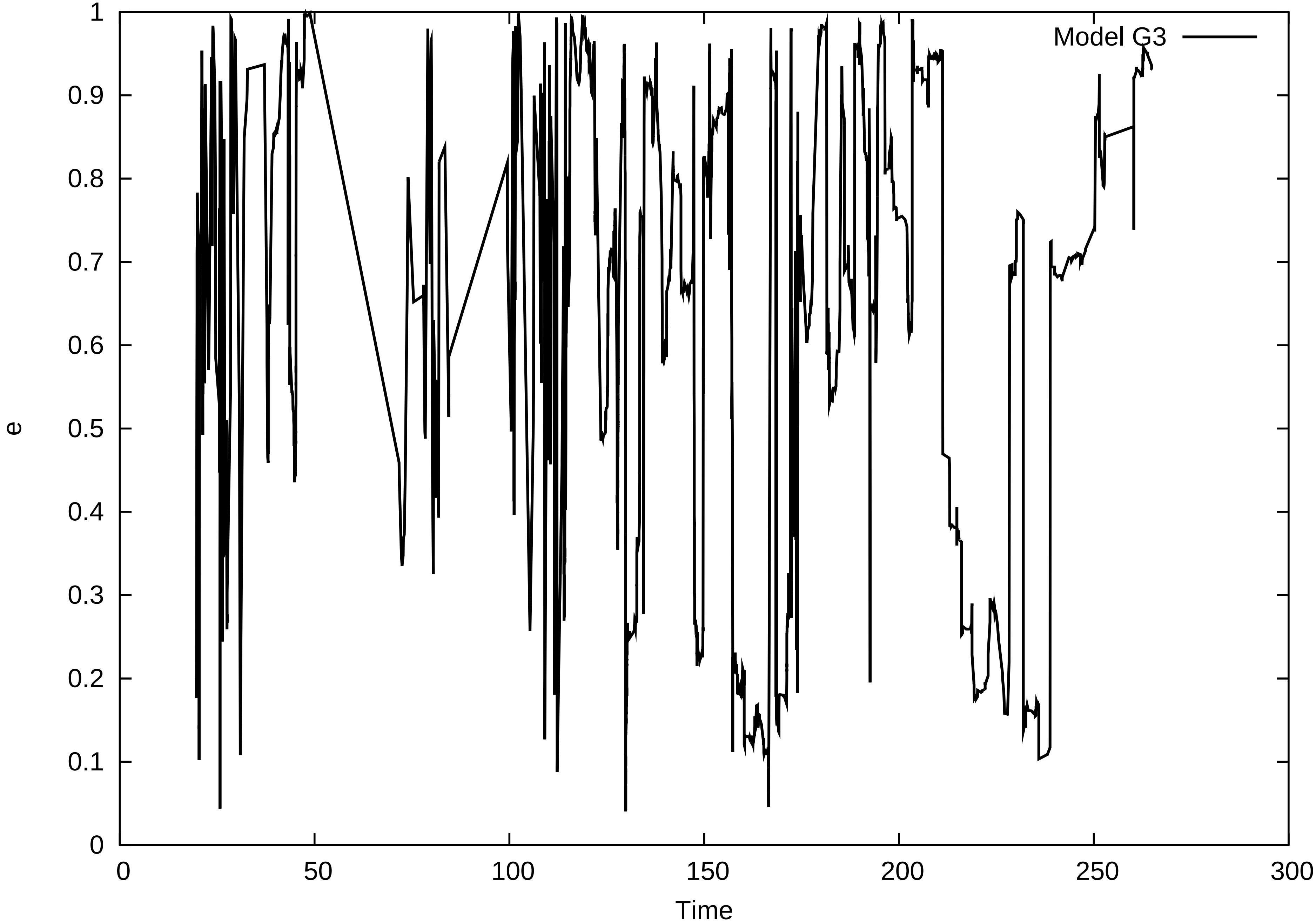
0.001
0.0001
1e-05
20
40
60
80
100
120
140
160
Time
a
Model G2



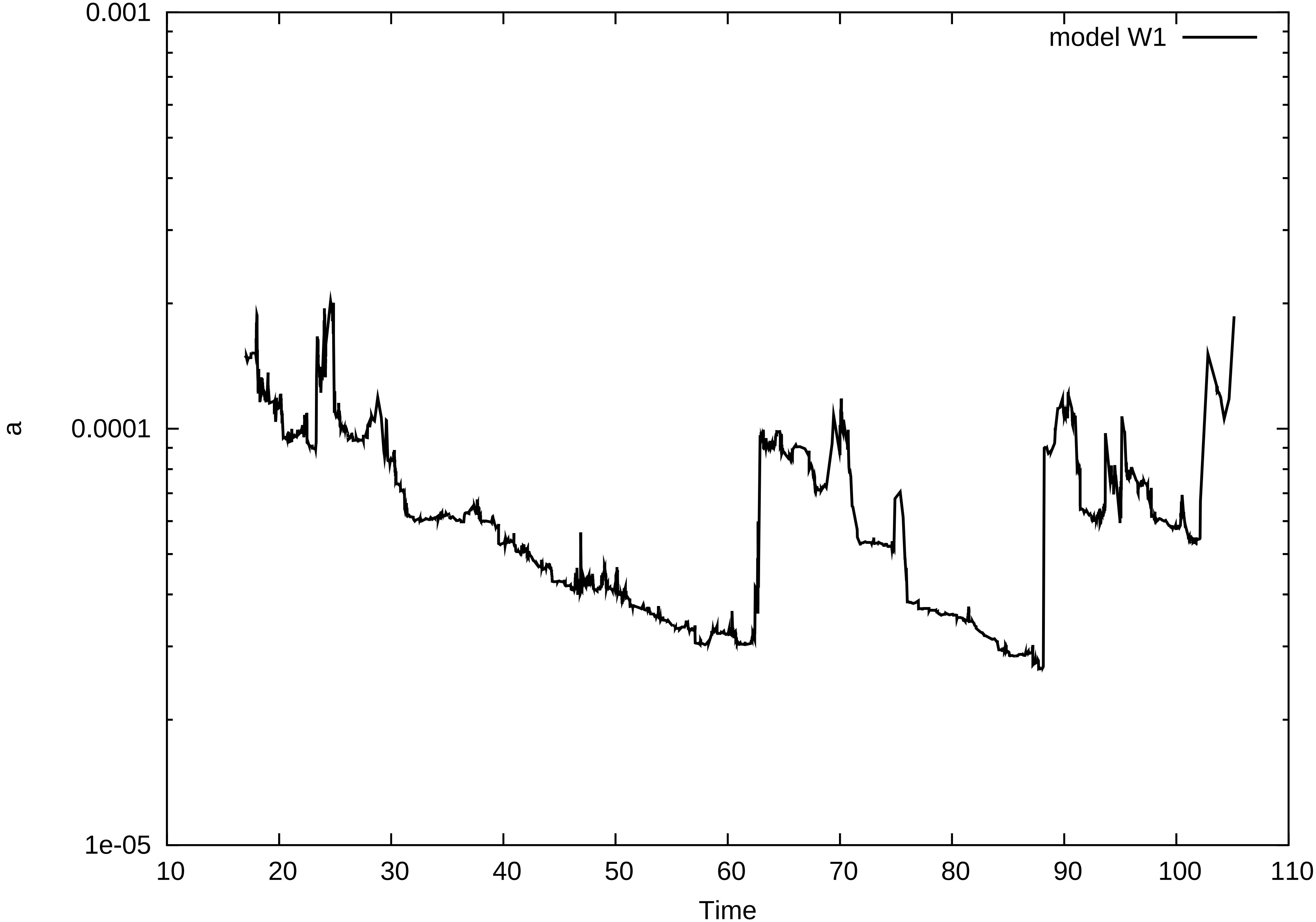
Model G2
e
Time
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
20
40
60
80
100
120
140
160

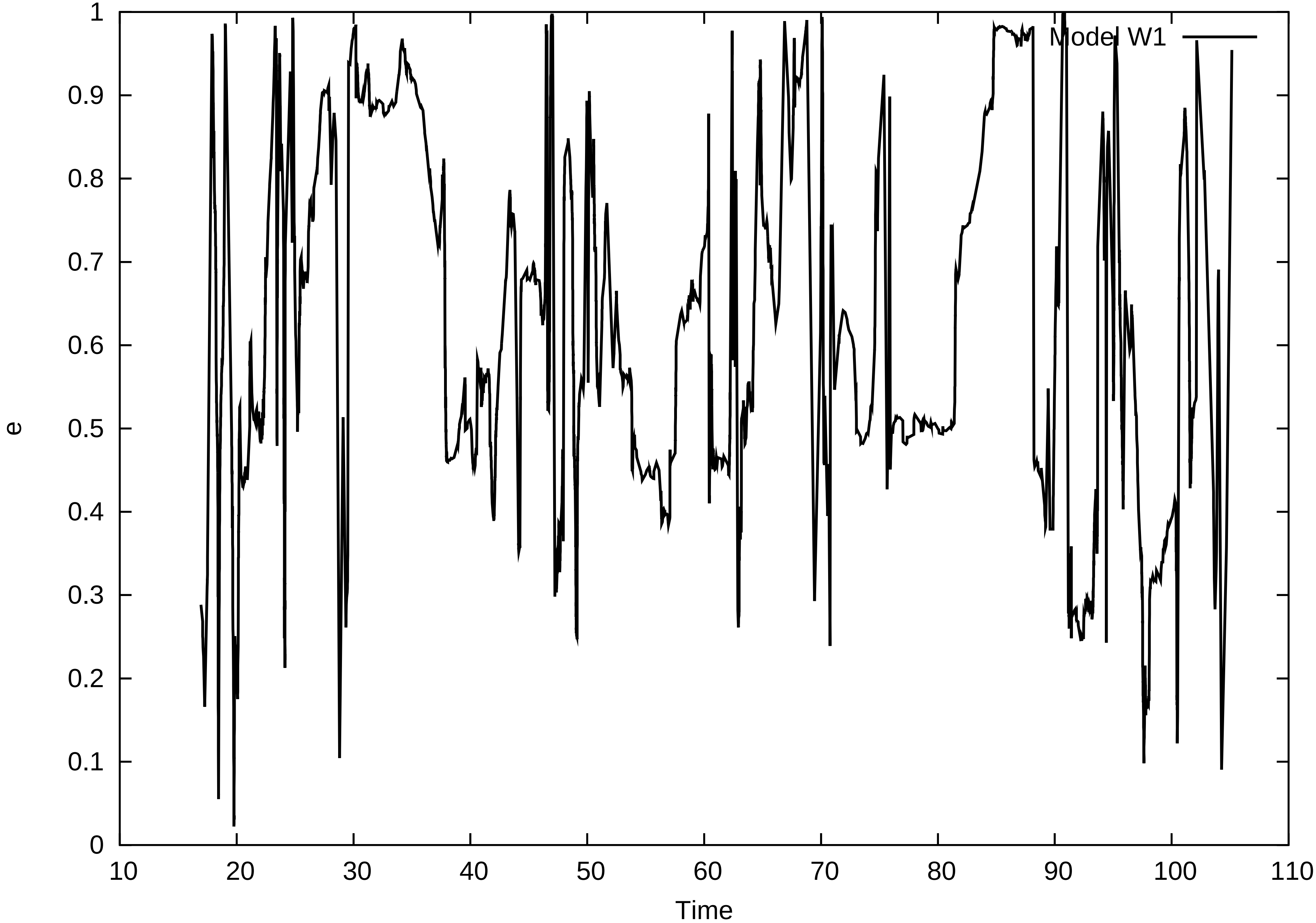


Model G3
0.01
0.001
0.0001
1e-05
1e-06
a
0
50
100
150
200
250
300
Time

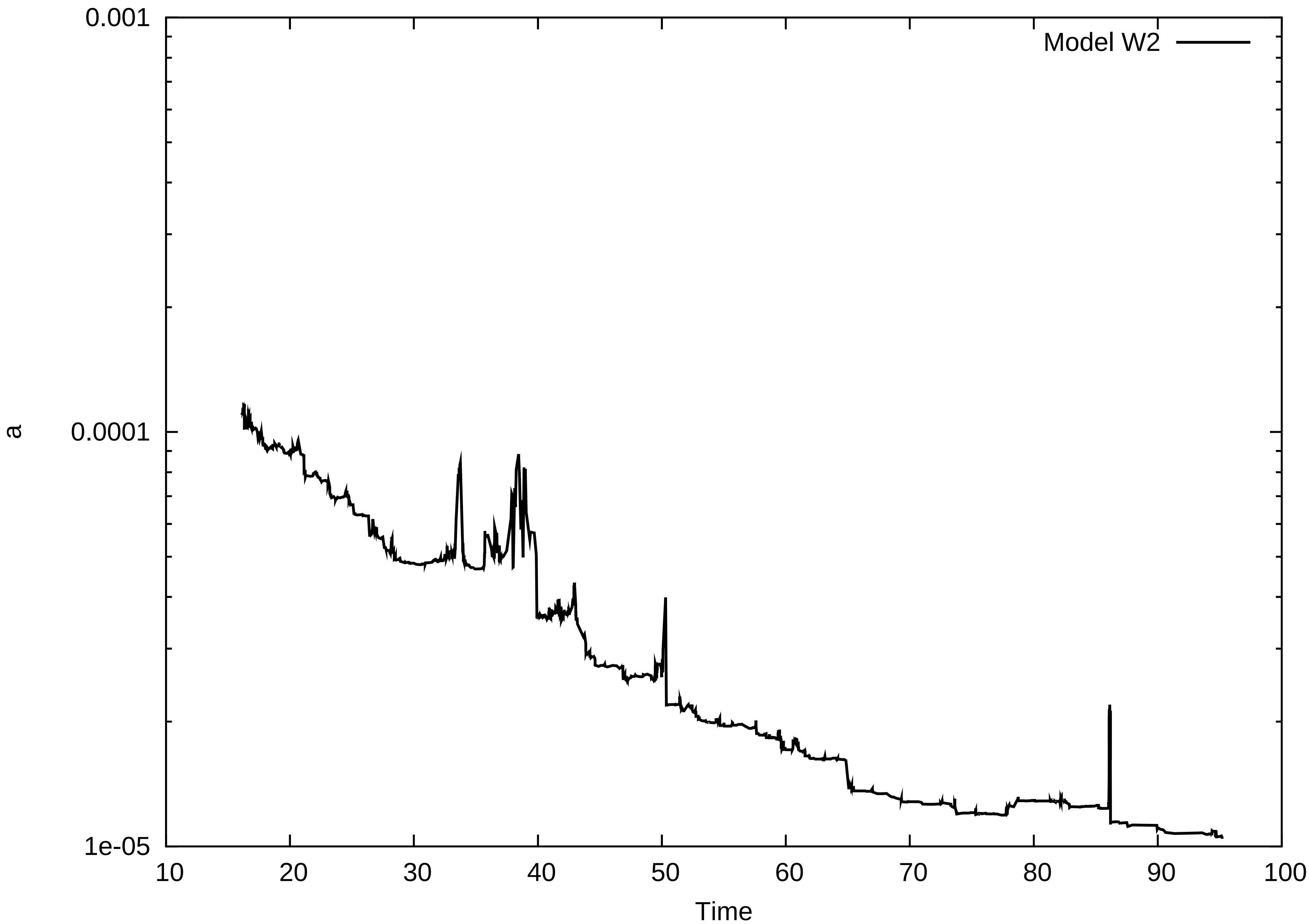


Model G3
e
Time
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
50
100
150
200
250
300

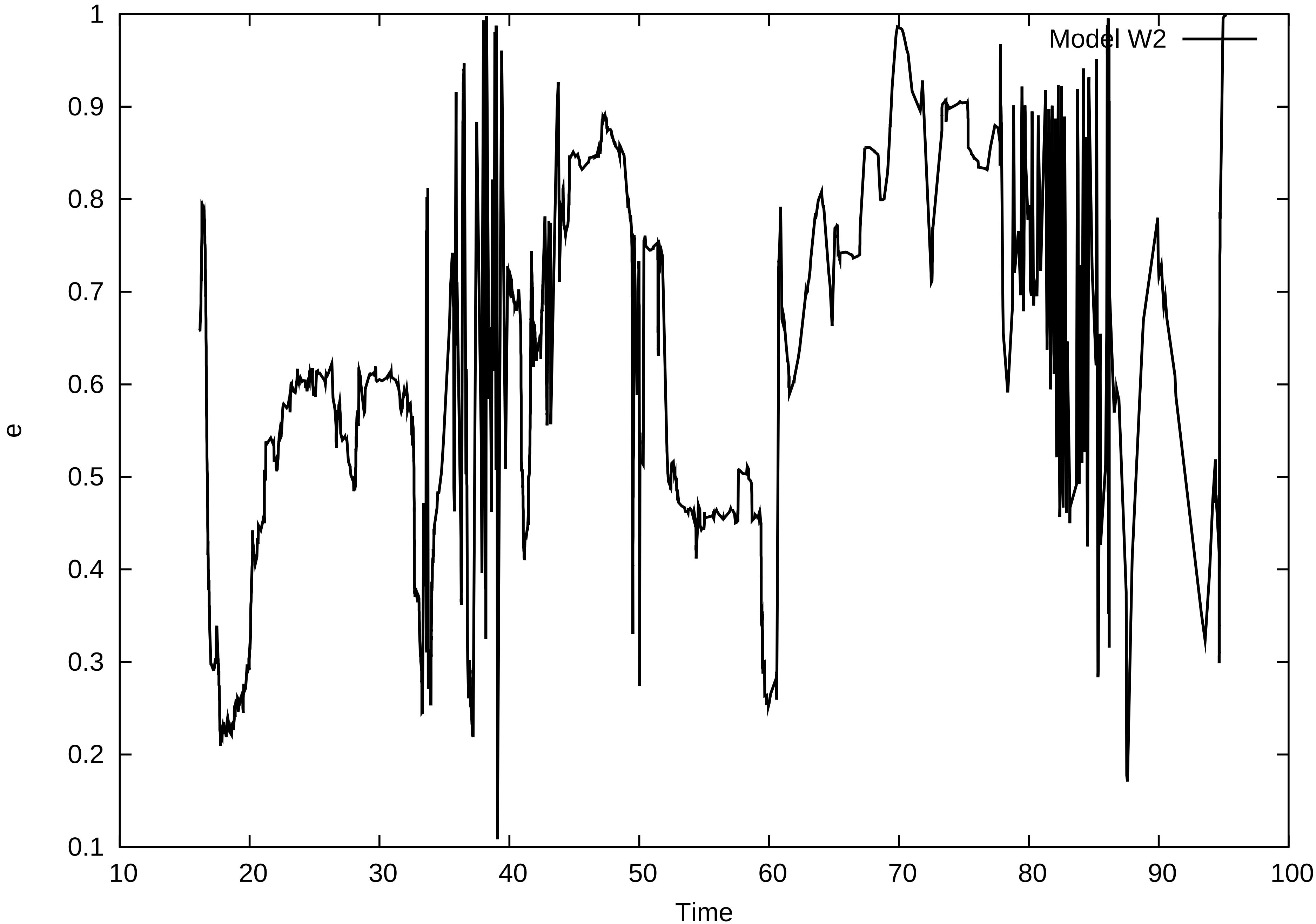

0.001
0.0001
1e-05
a
10
20
30
40
50
60
70
80
90
100
110
Time
model W1



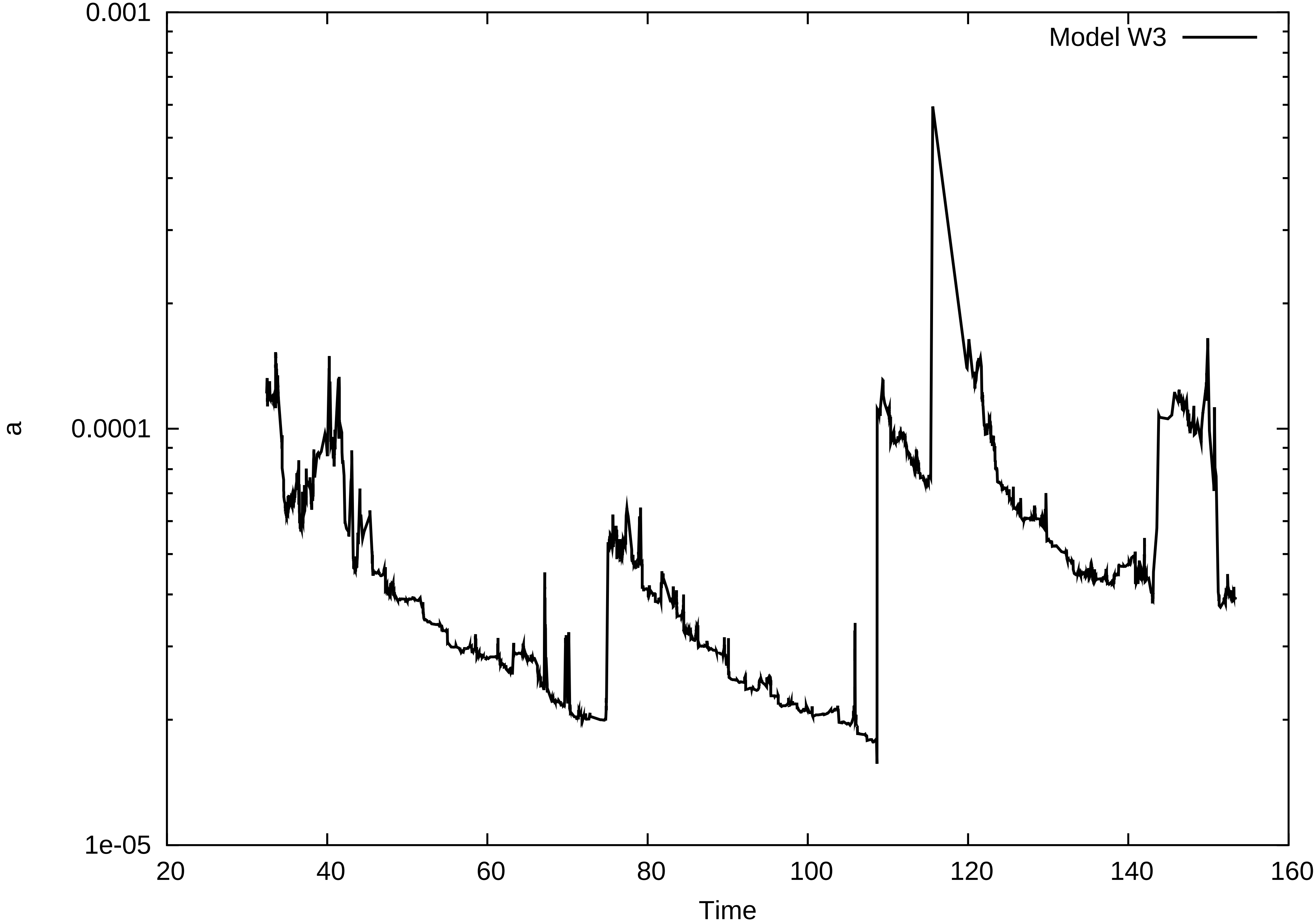
Model W1
e
Time
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
10
20
30
40
50
60
70
80
90
100
110



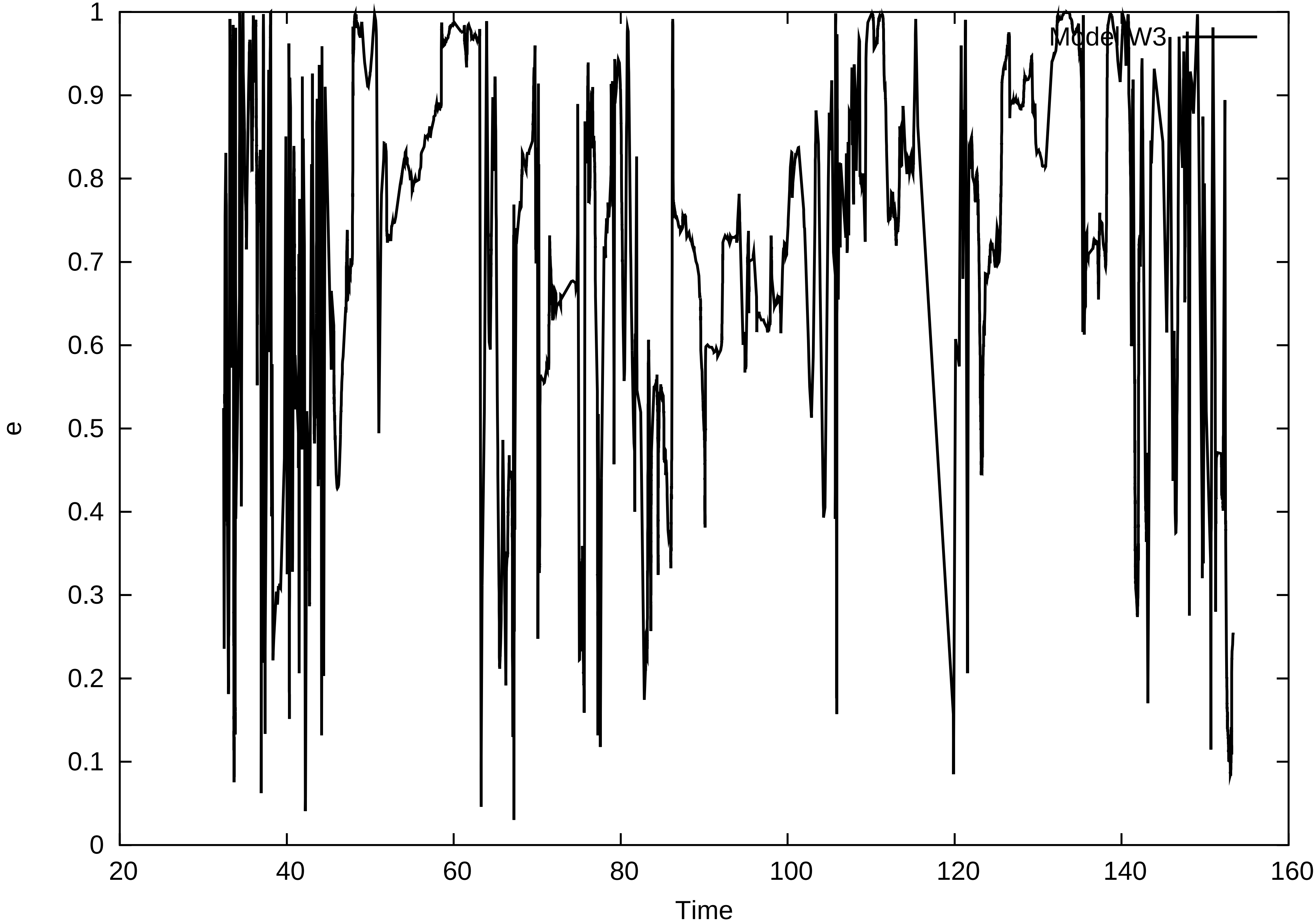
0.001
0.0001
1e-05
a
10
20
30
40
50
60
70
80
90
100
Time
Model W2



Model W2
e
Time
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
10
20
30
40
50
60
70
80
90
100



0.001
0.0001
1e-05
a
20
40
60
80
100
120
140
160
Time
Model W3


Model W3
e
Time
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
20
40
60
80
100
120
140
160