Stellar Motions in an Axially Perturbed Keplerian Potential

Ladislav Šubr

Astronomical Institute, Charles University, Prague

Kozai–Lidov mechanism

- evolution of a hiearchical triple system $M_1 > M_2 > M_3$

  Lidov 1961: Earth > Moon > satellite

  Kozai 1962: Sun > Jupiter > asteroid

- secular evolution of the orbital elements e, i and ω
- “averaging” technique of the Hamiltonian perturbation theory allows to get rid of “fast” variable (mean anomaly)
- integrals of motion: $a,\ c \equiv \sqrt{1-e^2}\,\cos i$ and $\bar{V}_{\rm d}$
- usable also for motion of a test particle in the compound field of the central mass and an axisymmetric perturbation (ring, torus, disc. . .)

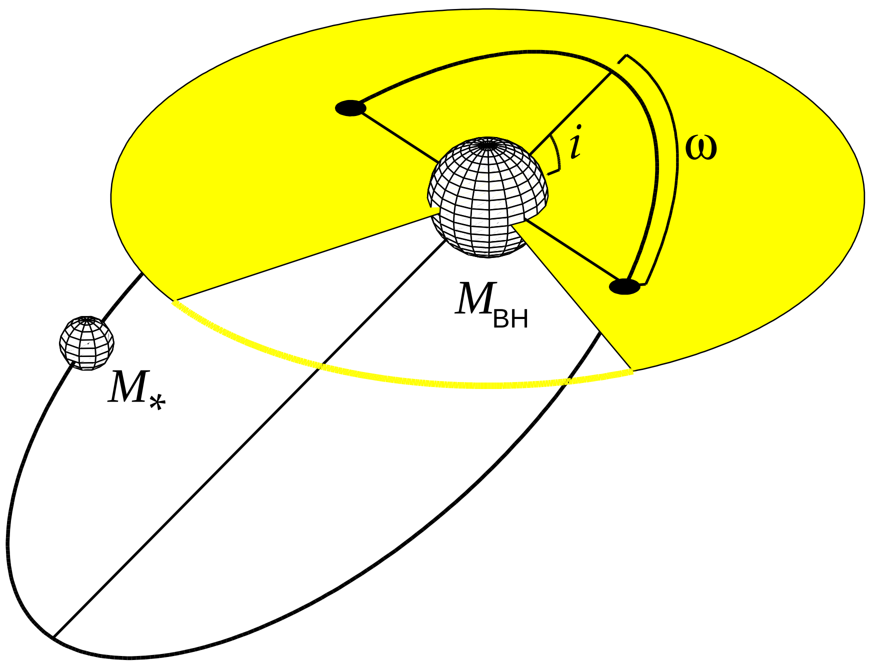
2-body Hamiltonian

Carthesian coordinates:

$$\mathcal{H} = \frac{1}{2}\left(v_1^2 + v_2^2 + v_3^2\right) - \frac{\mathcal{G}\,(m_0 + m_1)}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

Delaunay variables:

$$\mathcal{H} = -\frac{\mathcal{G}^2(m_0 + m_1)^2}{2L^2}$$



$$L = \sqrt{\mathcal{G}(m_0 + m_1)a} \qquad l = M$$

$$G = L\sqrt{1 - e^2} \qquad g = \omega$$

$$H = G\cos i \qquad h = \Omega$$

Sophus Lie

- Lie series:

$$\mathcal{S}_{\mathcal{H}}^{t} f \equiv f(0) + \sum_{1}^{\infty} \frac{t^n}{n!} \mathcal{L}_{\mathcal{H}}^{n} f \,, \quad \mathcal{L}_{\mathcal{H}}^{n} f \equiv \mathcal{L}_{\mathcal{H}}^{1} \mathcal{L}_{\mathcal{H}}^{n-1} f \,, \quad \mathcal{L}_{\mathcal{H}}^{1} f \equiv \{f, \mathcal{H}\} = \frac{\mathrm{d} f}{\mathrm{d} t}$$

- Lie's criterion:

Transformation $(p, q) \to (p', q')$ is canonical if $\exists$ *generating* Hamiltonian $\chi(p', q')$: $\dot{p}' = -\partial\chi/\partial q' \wedge \dot{q}' = \partial\chi/\partial p'$ and parameter ϵ:

$$p = p' + \int_0^{\epsilon} \dot{p}' \mathrm{d}t \equiv p'(\epsilon) \,, \quad q = q' + \int_0^{\epsilon} \dot{q}' \mathrm{d}t \equiv q'(\epsilon)$$

In terms of Lie series: $p = \mathcal{S}_{\chi}^{\epsilon} p' \,, \quad q = \mathcal{S}_{\chi}^{\epsilon} q'$

$$\mathcal{H}(p,q) = \mathcal{H}_0(p) + \epsilon\mathcal{H}_1(p,q)$$

(?) Canonical transformation $p = p' + \epsilon f(p', q')\,,\; q = q' + \epsilon g(p', q')$:

$$\mathcal{H}(p,q) \to \mathcal{H}'(p',q') = \mathcal{H}_0(p') + \epsilon\bar{\mathcal{H}}_1(p') + \epsilon^2\mathcal{H}_2(p',q')$$

$$p = \mathcal{S}_\chi^\epsilon p'\ ,\quad q = \mathcal{S}_\chi^\epsilon q'$$

$$\mathcal{H}(p,q) = \mathcal{H}\left(p'(\epsilon), q'(\epsilon)\right) \equiv \mathcal{H}'(p',q') = \mathcal{S}_\chi^\epsilon\mathcal{H}\left(p'(0), q'(0)\right)$$

$$\mathcal{H}' = \mathcal{H}_0 + \epsilon\mathcal{H}_1 + \epsilon\{\mathcal{H}_0,\chi\} + \epsilon^2\{\mathcal{H}_1,\chi\} + \frac{\epsilon^2}{2}\{\{\mathcal{H}_0,\chi\},\chi\} + \mathcal{O}(\epsilon^3)$$

(?) $\exists\, \chi(p',q'),\ \bar{\mathcal{H}}_1(p') :\ \mathcal{H}_1 + \{\mathcal{H}_0,\chi\} = \bar{\mathcal{H}}_1$

Yes

q' are angles and Hamiltonian $\mathcal{H}$ is periodic in $q' \longrightarrow$ Fourier:

$$\mathcal{H}_1(p', q') = \sum_{k \in \mathrm{Z}^n} c_k(p') \exp[i\, k.q']$$

…and look for χ in the same form:

$$\chi(p', q') = \sum_{k \in \mathrm{Z}^n} d_k(p') \exp[i\, k.q']$$

$$\{\mathcal{H}_0, \chi\} = -i \sum_{k \in \mathrm{Z}^n} d_k(p')\, k.\omega_0(p') \exp[i\, k.q'] \ , \quad \omega_0 = \nabla_{p'} \mathcal{H}_0$$

solution:

$$d_0 = 0 \ , \quad d_k(p') = -i \frac{c_k(p')}{k.\omega_0(p')} \ \ \forall k \neq 0 \ , \quad \bar{\mathcal{H}}_1(p') = c_0(p')$$

3-body problem

$$\mathcal{H} = \mathcal{H}_{\mathrm{K}}(p^1, q^1) + \mathcal{H}_{\mathrm{K}}(p^2, q^2) + \frac{\mathcal{G}m_2(m_0 + m_1)}{|r_2|} - \frac{\mathcal{G}m_0 m_2}{|r_2|} - \frac{\mathcal{G}m_1 m_2}{|r_{12}|}$$

$$\mathcal{H}_1 = -\frac{\mu}{2}\frac{|r_{01}|^2}{|r_2|^3}\left(\frac{3r_{01}.r_2}{|r_{01}||r_2|} - 1\right) + \frac{\mu}{|r_2|}\mathcal{O}\left(\frac{|r_{01}|^3}{|r_2|^3}\right) , \quad \mu \equiv \frac{\mathcal{G}m_0 m_1 m_2}{m_0 + m_1}$$

$$\bar{\mathcal{H}} = -\frac{\mu a_1^2}{8a_2^3(1 - e_2^2)^{3/2}}\left((2 + 3e_1^2)(3\cos^2 I - 1) + 15e_1^2 \sin^2 I \cos 2\omega\right)$$

Kozai equations

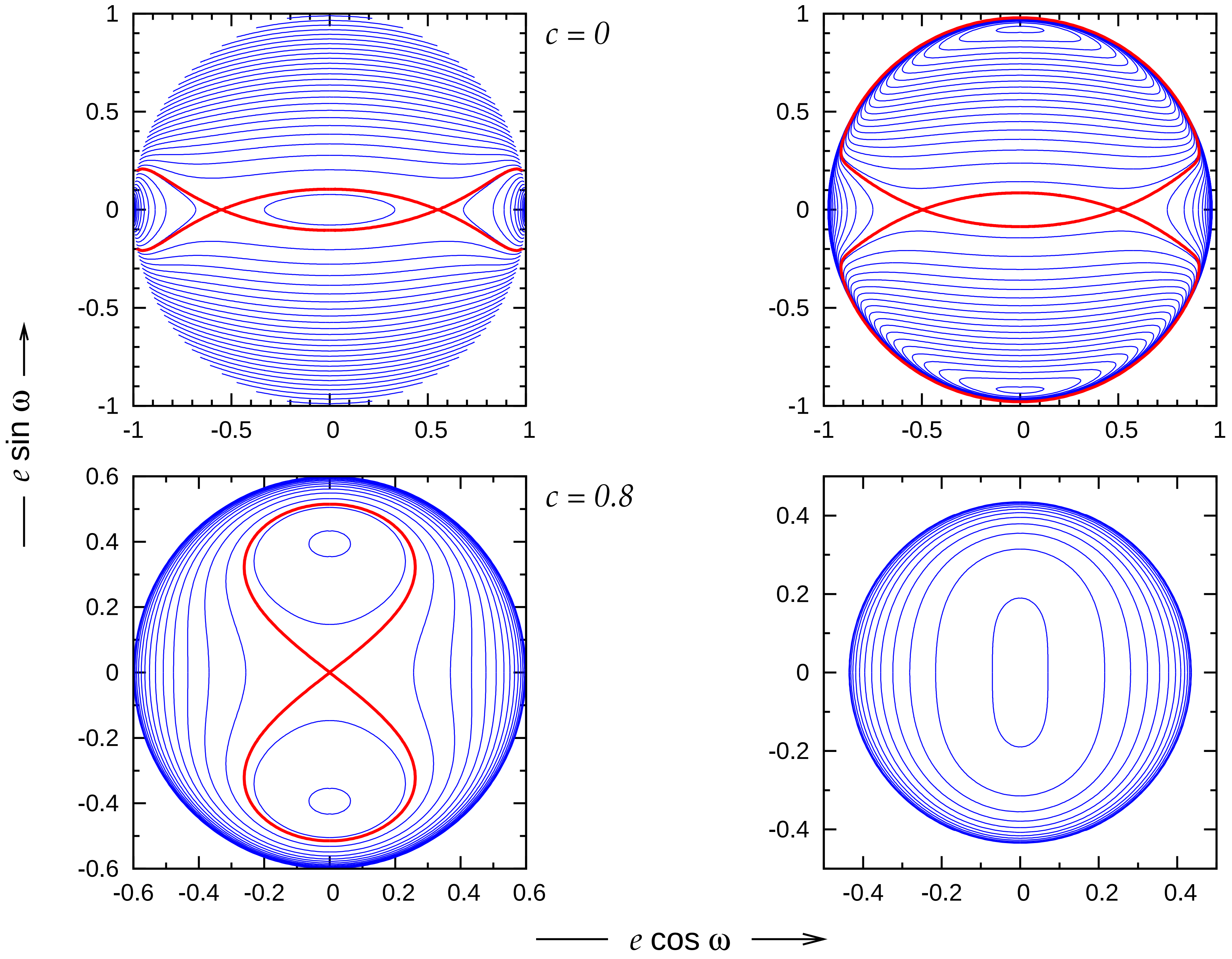
$$T_K \sqrt{1-e^2}\,\frac{\mathrm{d}i}{\mathrm{d}t} = -5e^2 \sin i\, \cos i\, \sin\omega\, \cos\omega$$

$$T_K \sqrt{1-e^2}\,\frac{\mathrm{d}e}{\mathrm{d}t} = 5e(1-e^2)\sin^2 i\, \sin\omega\, \cos\omega$$

$$T_K \sqrt{1-e^2}\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = 2(1-e^2) + 5(e^2 - \sin^2 i)\sin^2\omega$$

$$T_K \equiv \frac{4}{3}\frac{M_{BH}}{M_d}\left(\frac{R_d}{a}\right)^3 P$$

$\bar{V}_\mathrm{d}$ = const. *contours*



Effect of the relativistic pericentre advance

- characteristic time–scales:

$$T_{\rm K} = \frac{4}{3}\frac{M_{\rm BH}}{M_{\rm d}}\left(\frac{R_{\rm d}}{a}\right)^3 P \qquad \text{vs.} \qquad T_{\rm E} = \frac{1}{3}\frac{a(1-e^2)}{R_{\rm g}} P$$

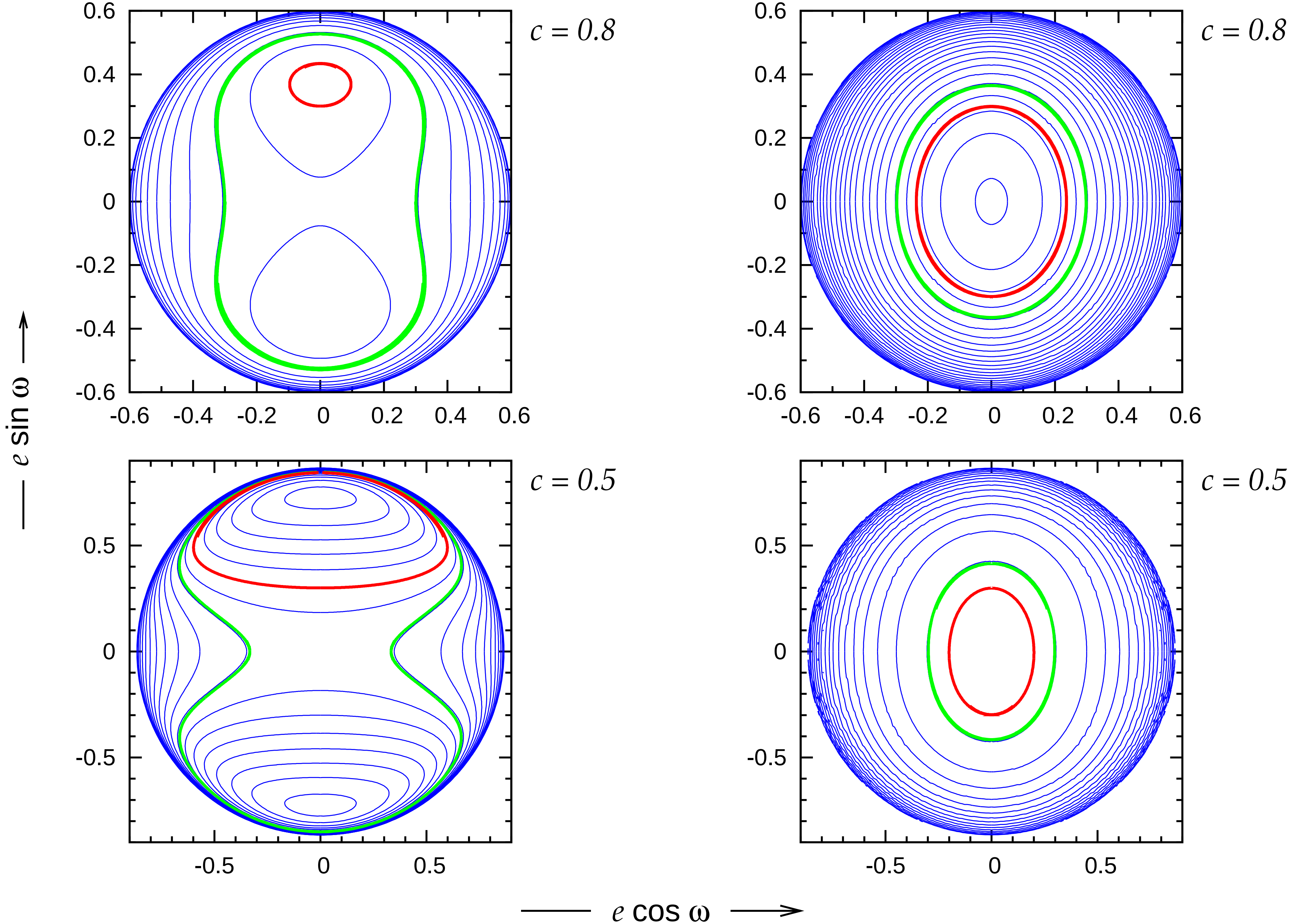
Kozai wins for $a > a_{\rm min} \approx \left(\frac{M_{\rm BH}}{M_{\rm d}}\right) R_{\rm d}^{6/7} R_{\rm g}^{2/7} R_{\rm p}^{-1/7}$

- perturbations to the Keplerian potential:

$$V_{\rm p} = -\frac{GM_{\rm BH}}{r-2R_{\rm g}} = -\frac{GM_{\rm BH}}{r} - \frac{2GM_{\rm BH}}{r(r-2R_{\rm g})}$$

$$V_{\rm d} = -\frac{2GM_{\rm d}}{\pi}\frac{K(k)}{B}\,, \quad B^2 \equiv z^2 + (r+R_{\rm d})^2\,, \quad k^2 \equiv \frac{4rR_{\rm d}}{B^2}$$

Effect of the relativistic pericentre advance II



Tidal disruptions

- enhanced tidal disruptions due to the eccentricity oscillations
- supply of gas for accretion discs; food for black holes

$$R_{\rm t} = \left(\frac{M_{\rm BH}}{M_*}\right)^{1/3} R_* = 47 \left(\frac{M_{\rm BH}}{10^6 M_\odot}\right)^{-2/3} \left(\frac{M_*}{M_\odot}\right)^{-1/3} \left(\frac{R_*}{R_\odot}\right) R_{\rm g}$$

- characteristic radius of the stellar cluster $\sim 10^6 R_{\rm g} \Longrightarrow$ extreme eccentricities
- $\mathcal{F}(r_{\rm min})$: fraction of stars from an enssemble with given (initial) distribution of orbital elements $n(a, e, i, \omega)$ that pass the centre within $r_{\rm min}$ at some moment.

Trivial case

- cluster dominated by the gravity of the central mass
- distribution function: $n(a, e, i, \omega) \propto a^{1/4}\, e\, \sin i$

$$\mathcal{F}(a, c, r_{\rm min}) = F_0\, a^{1/4} \left(\sqrt{1 - e_{\rm min}^2} - c \right)$$

$$e_{\rm min} \equiv 1 - r_{\rm min}/a\,, \quad F_0 = \frac{5}{4} \left(a_{\rm max}^{5/4} - a_{\rm min}^{5/4} \right)^{-1}$$

$$\mathcal{F}(r_{\rm min}) \approx 5 r_{\rm min}\, \frac{a_{\rm max}^{1/4} - a_{\rm min}^{1/4}}{a_{\rm max}^{5/4} - a_{\rm min}^{5/4}}$$

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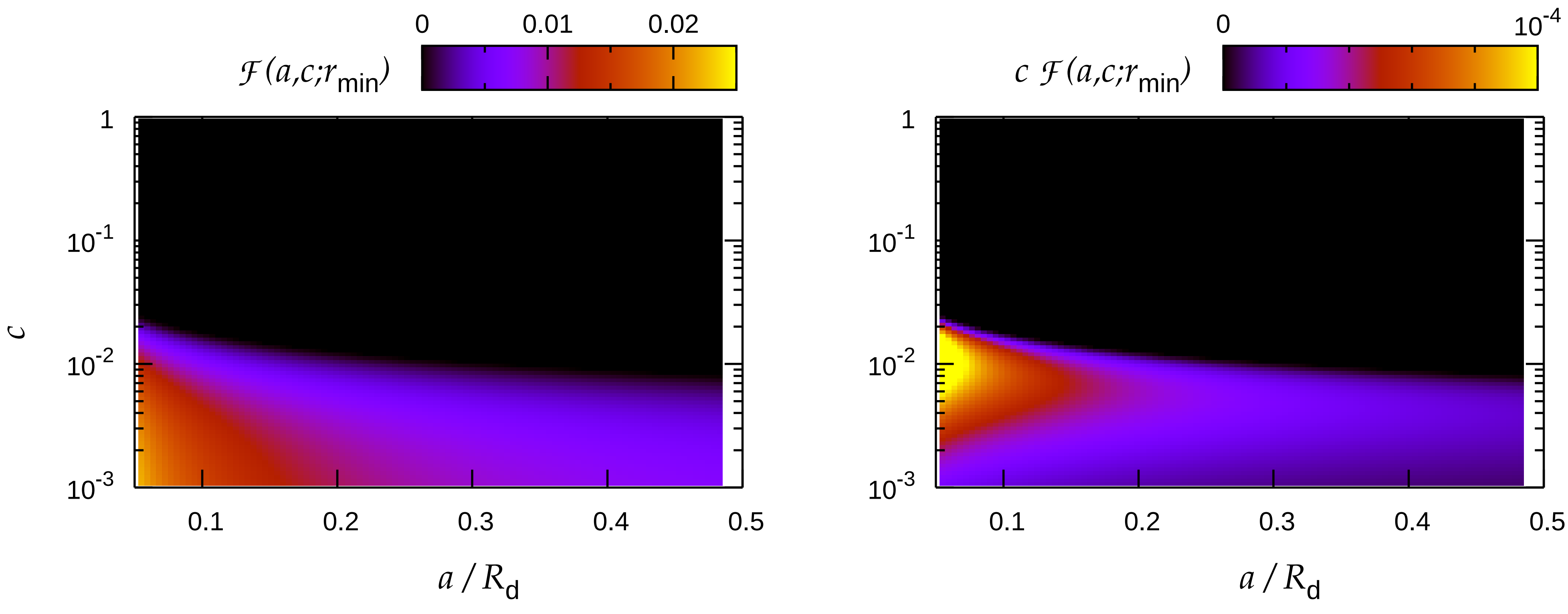
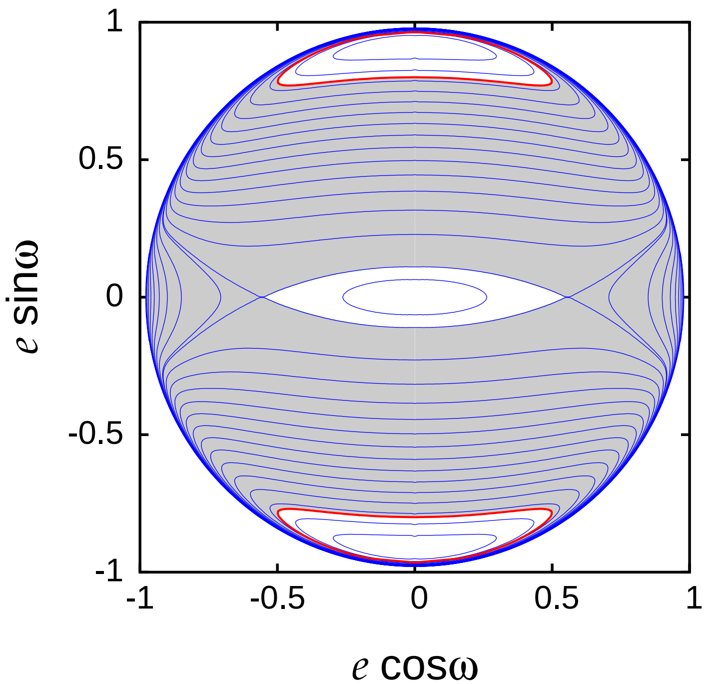
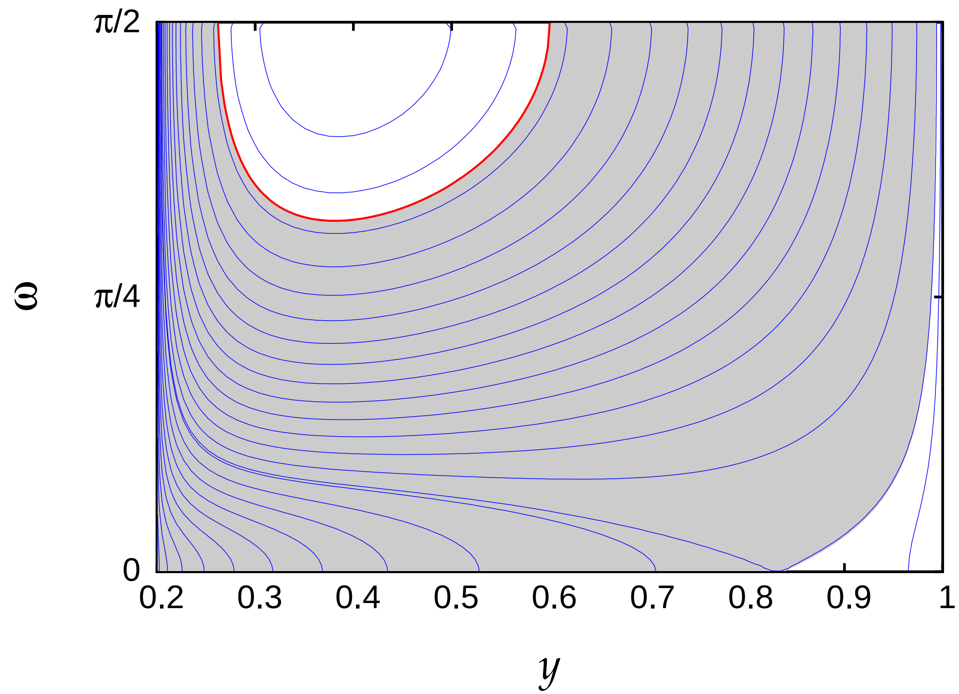


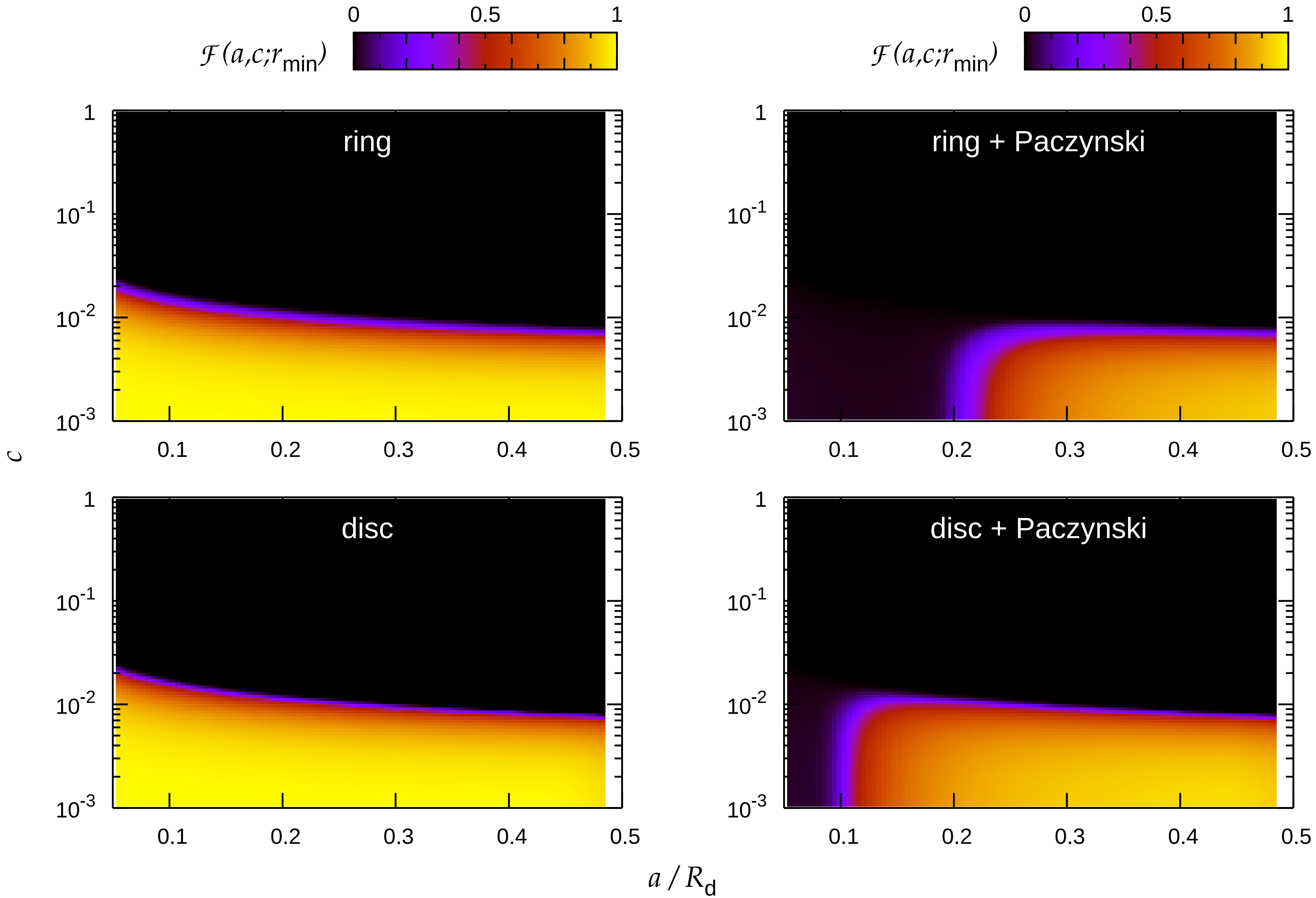
Less trivial case

- gravity of central mass plus axisymmetric perturbation
- doable with knowledge of the topology of $\bar{V}_\mathrm{d}$ contours in the (e, ω) space: maximum eccentricity reached at $\omega = \pi/2$





Semi-results. . .



... *integrated*

- ring-like perturbation:

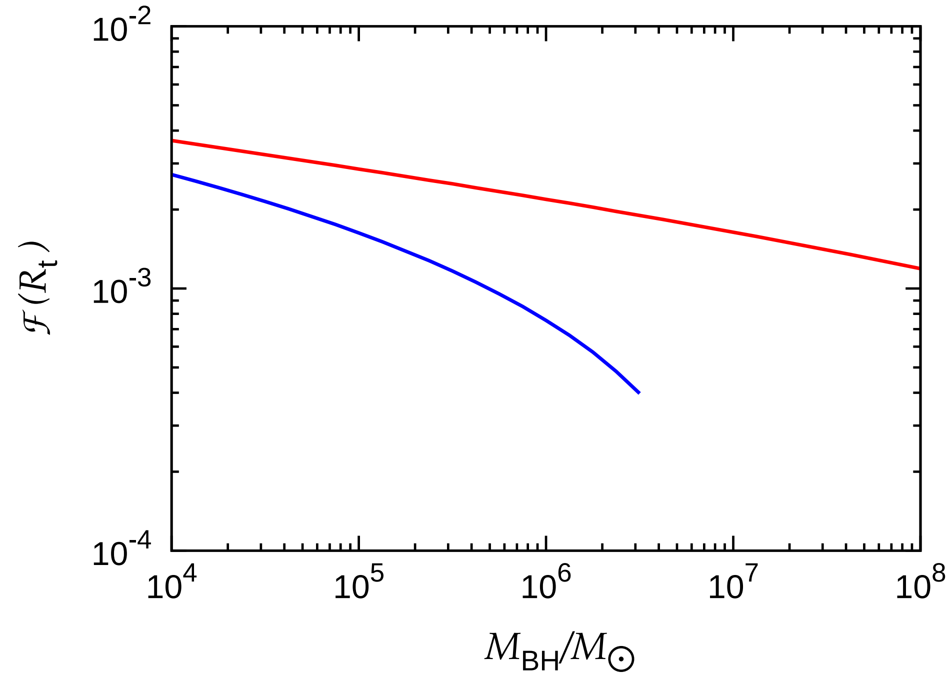
  $M_{\rm d} = 0.01 M_{\rm BH}$

  $R_{\rm d} = R_{\rm h} \approx 2.3 \times 10^6\, M_8^{-1/2}$

  $a_{\rm max} = 0.45 R_{\rm d}$

  $a_{\rm min} = 0.1 a_{\rm max}$

  $r_{\rm min} = R_{\rm t}$



- see also

  Ivanov et al., 2005, MNRAS, 358, 1361

  Hopman et al., 200x

S2 et al.

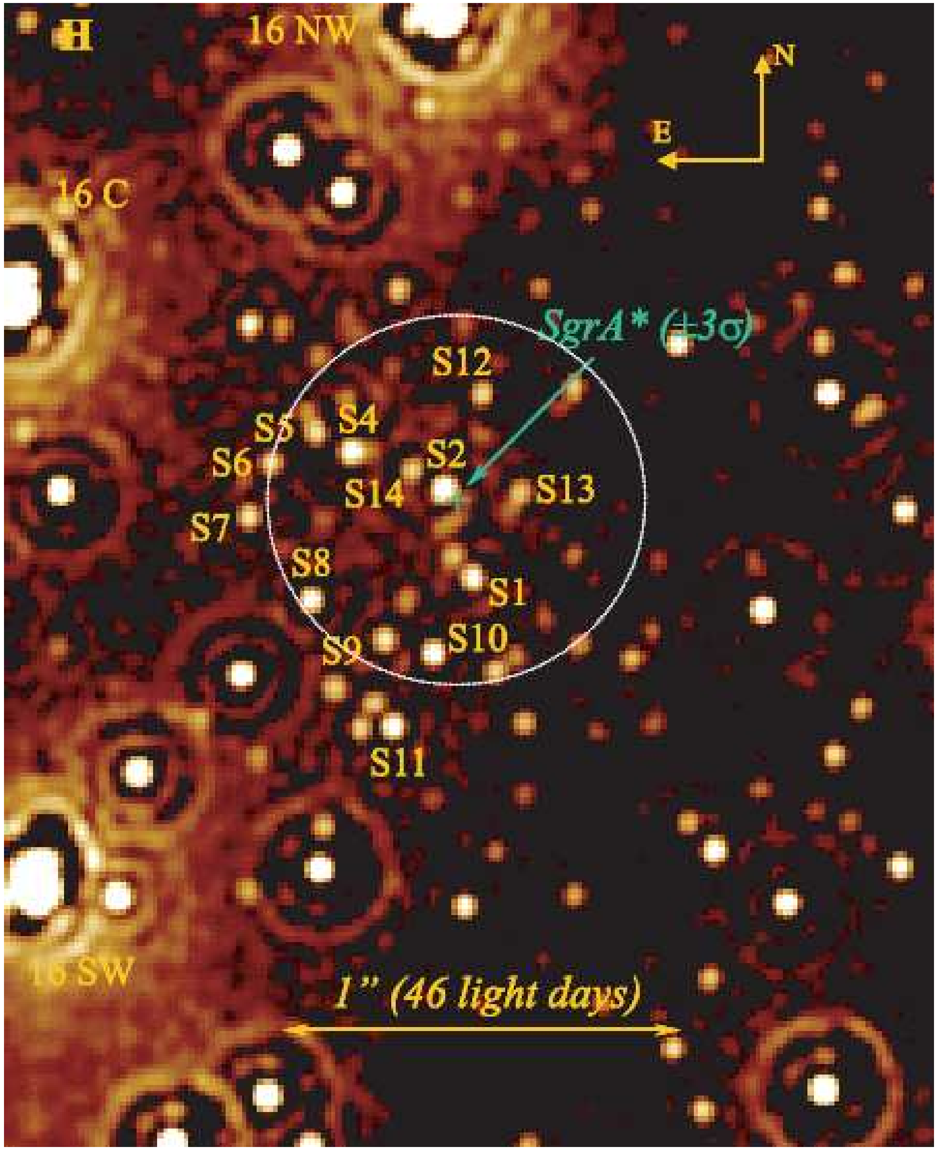
Classification:

- O9 main-sequence star
- $M_* \approx 15 M_\odot$
- age $\lesssim 10\mathrm{Myr}$

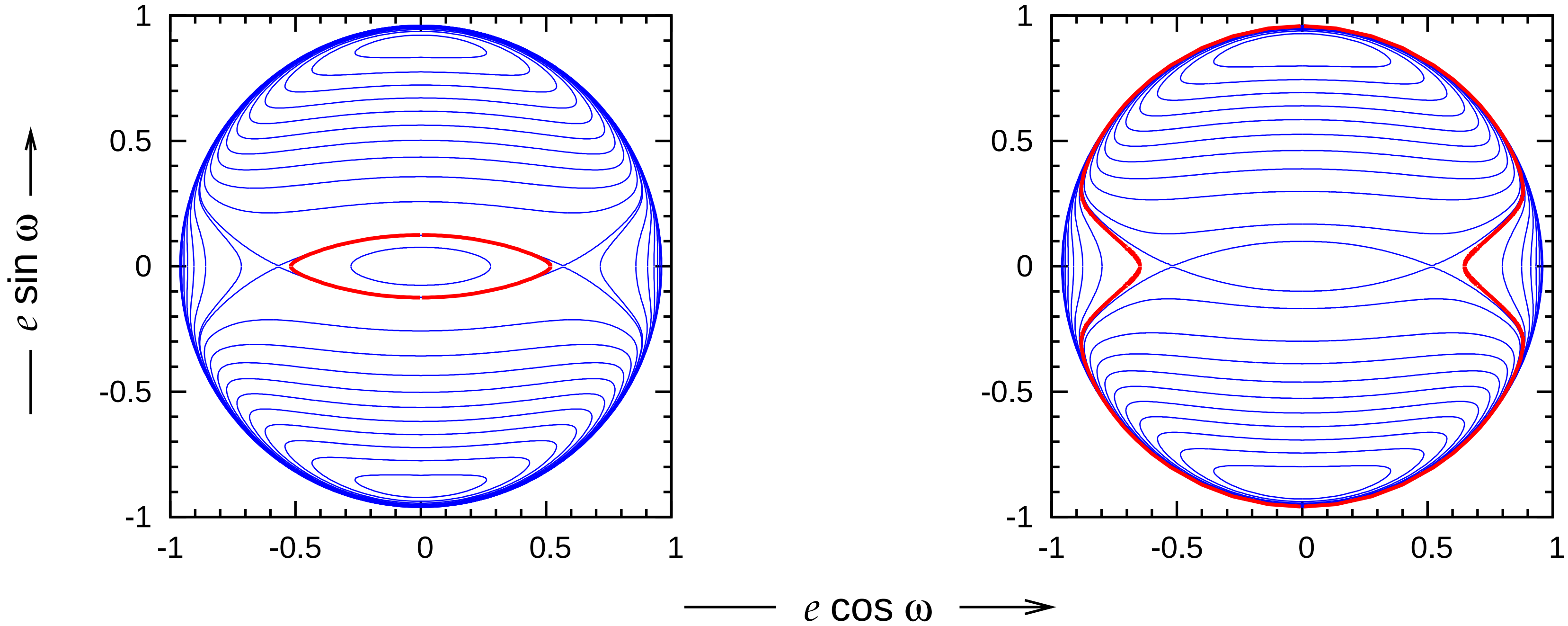
Kinematics:

- $P = 15.8\,\mathrm{yr}$
- $a = 2.5 \times 10^4 R_\mathrm{g}$
- $e = 0.87$
- $i = -47.3^\circ$

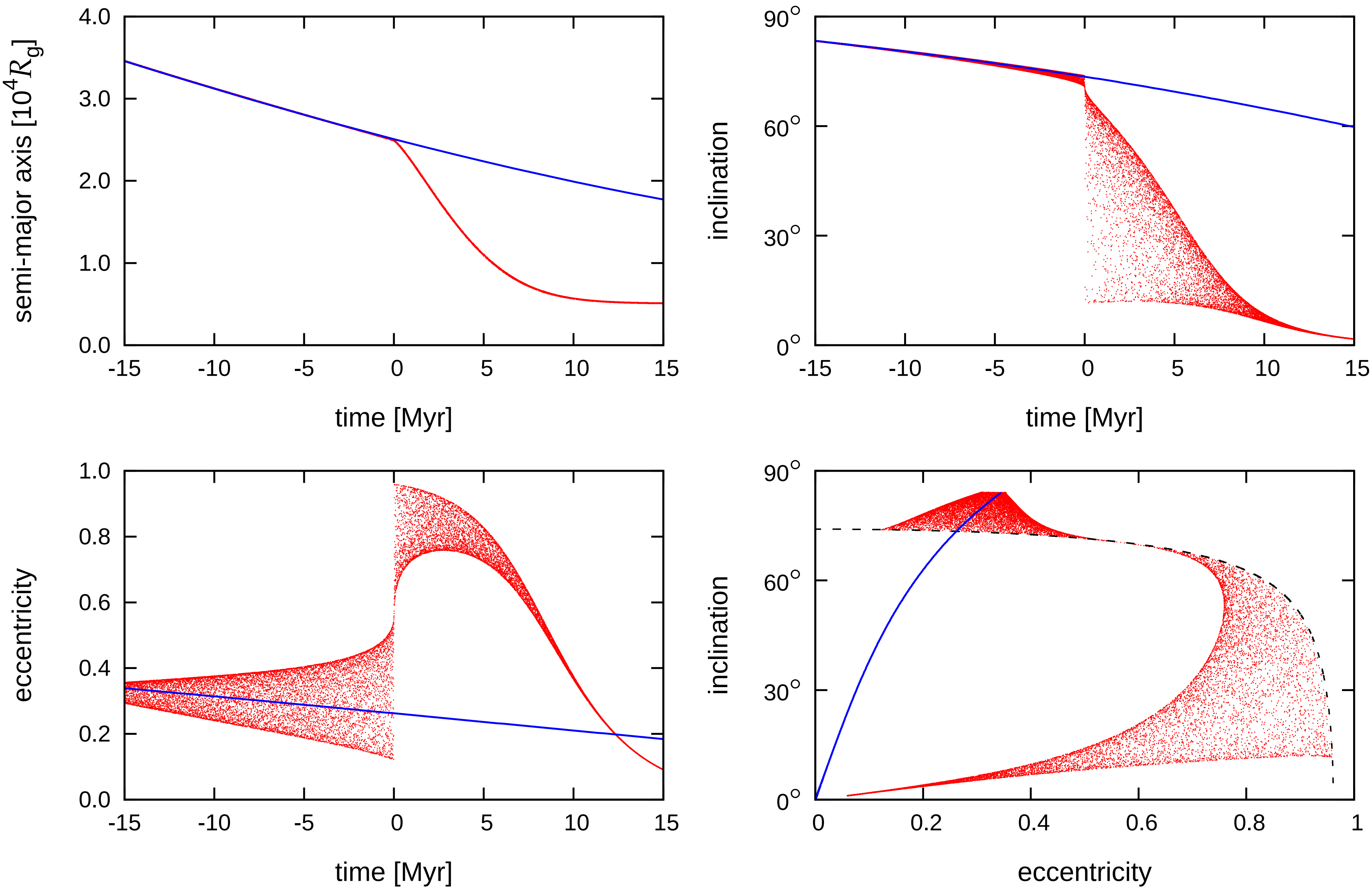
Puzzle: Too young to be there



Key point — the Kozai resonance



Temporal evolution



Temporal evolution II