astro8405

An Introduction to the
Cosmic Microwave Background

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eCampus | Lernplattform der Universität Bonn

astro8405: The Cosmic Microwave Background

Aktionen

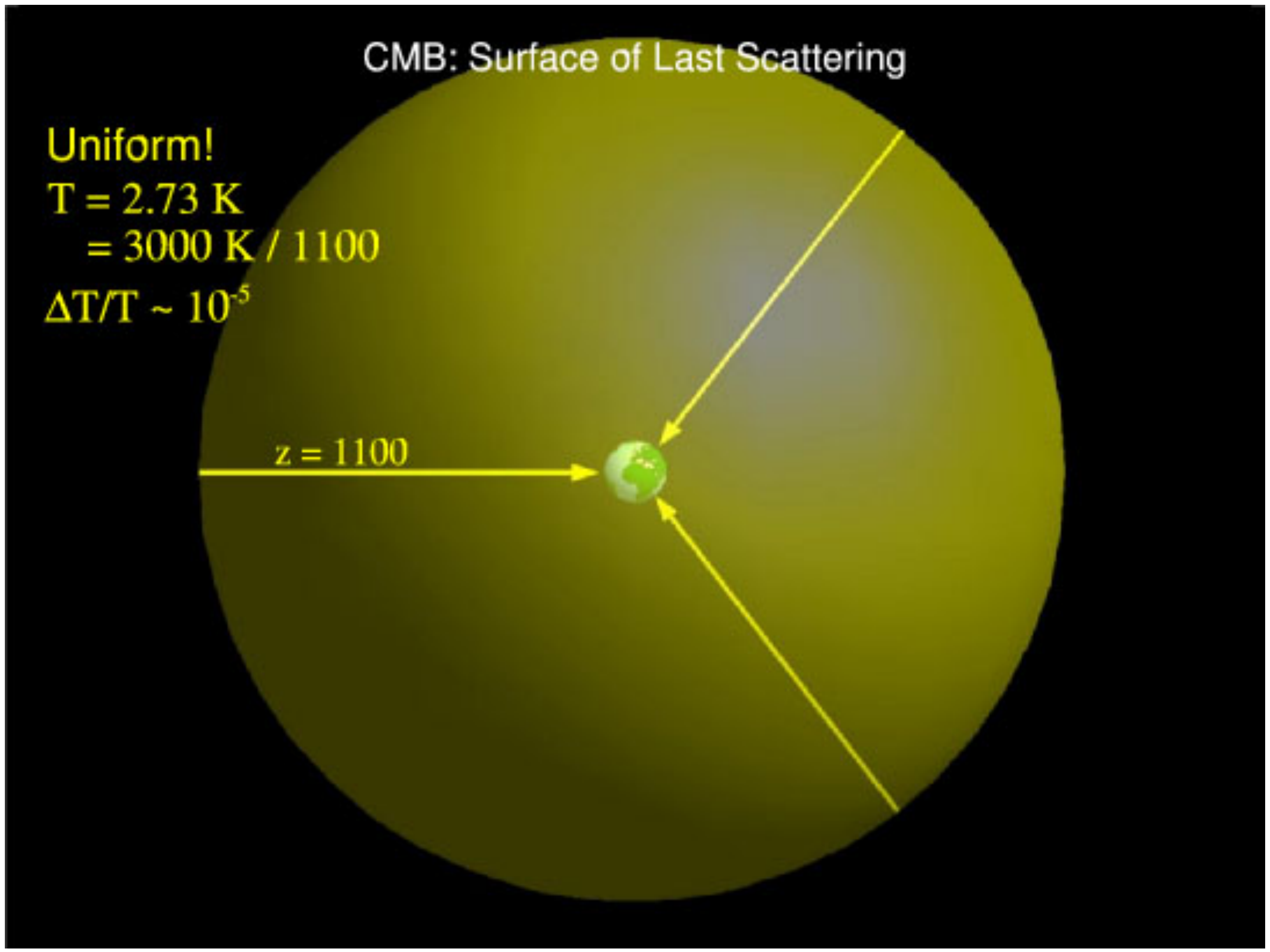
This course intends to give you a modern and up-to-date introduction to the science and experimental techniques relating to the Cosmic Microwave Background. No prior knowledge of cosmology is necessary, your prerequisite are a basic understanding of electrodynamics and thermal physics and some familiarity with Python programming.

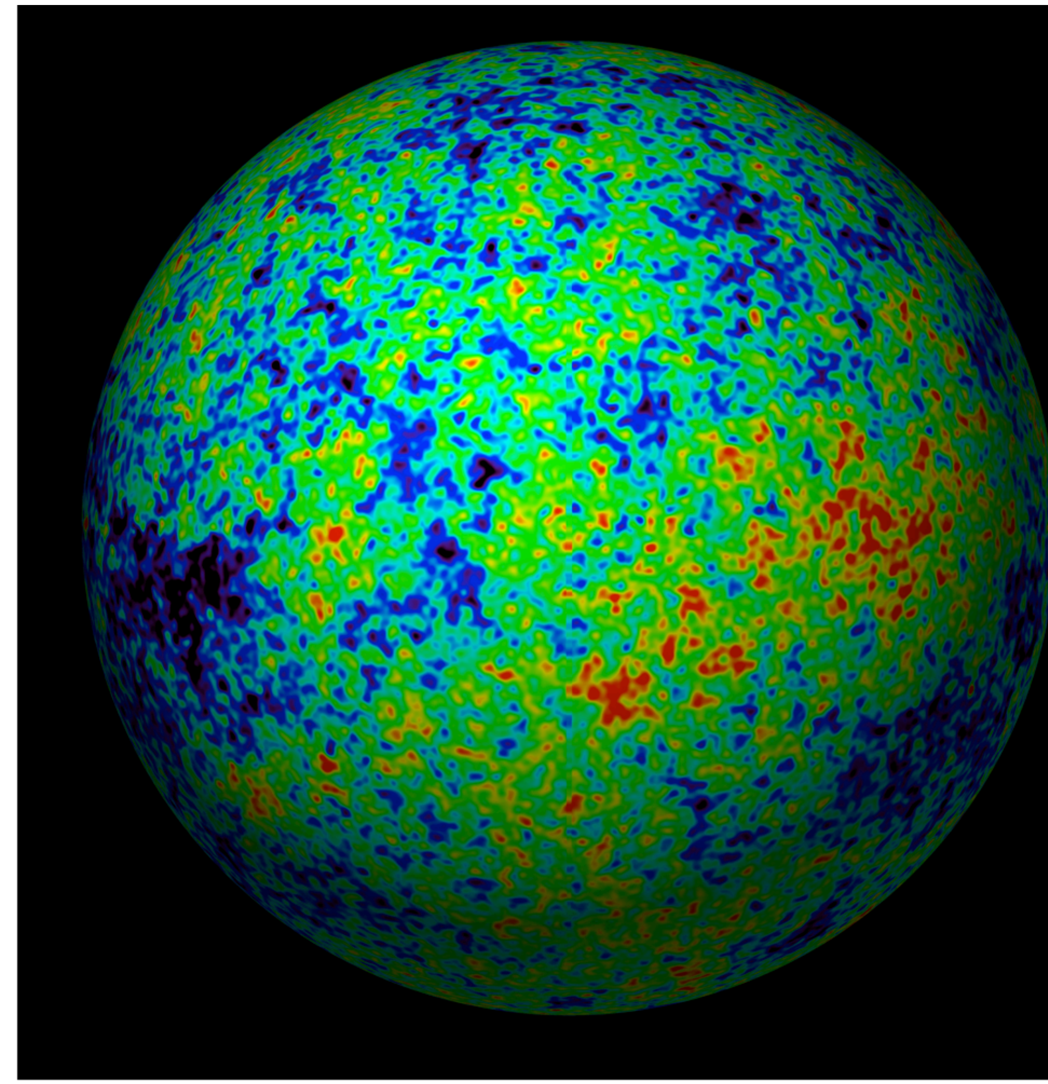
Lecture 4:

Temperature Anisotropies

The Angular Power Spectrum

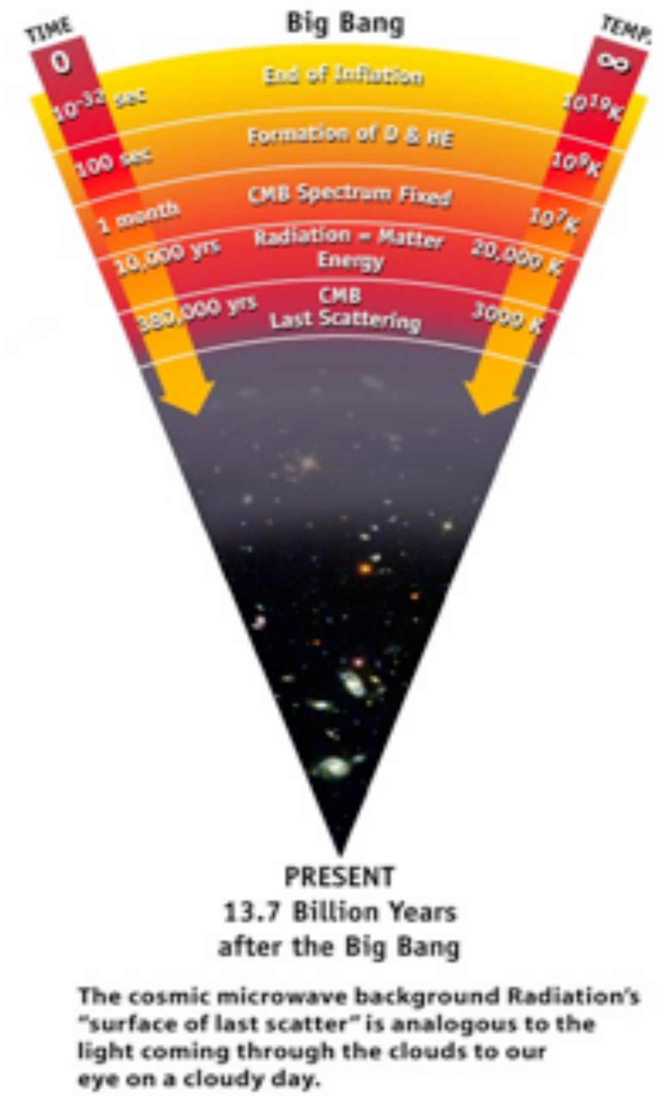
The last scattering “sphere”

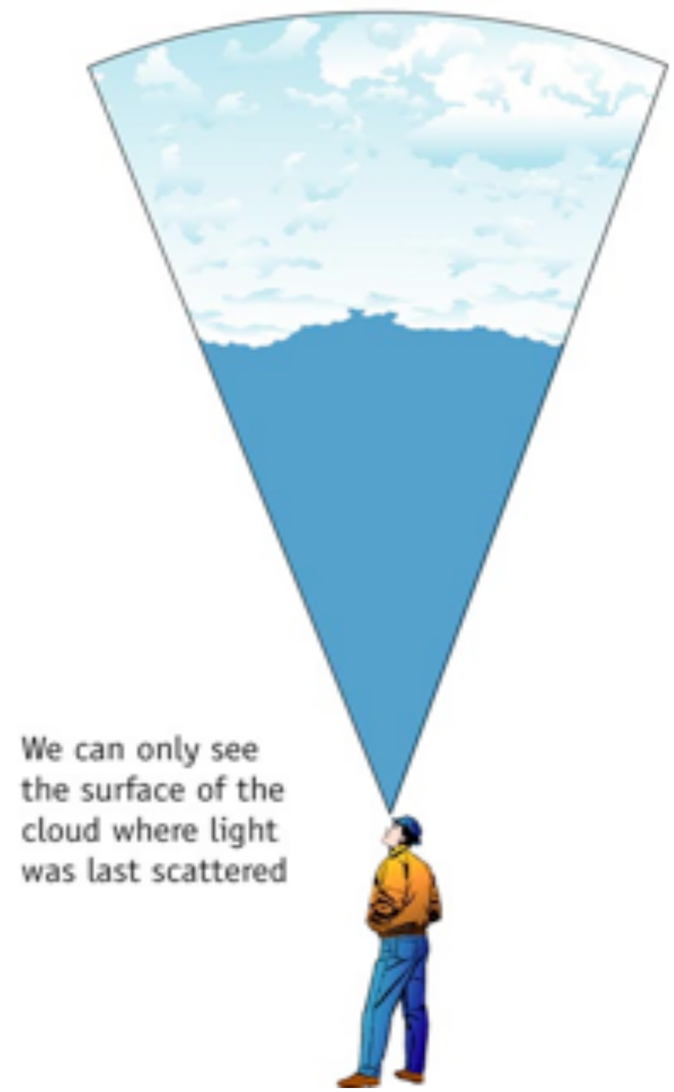


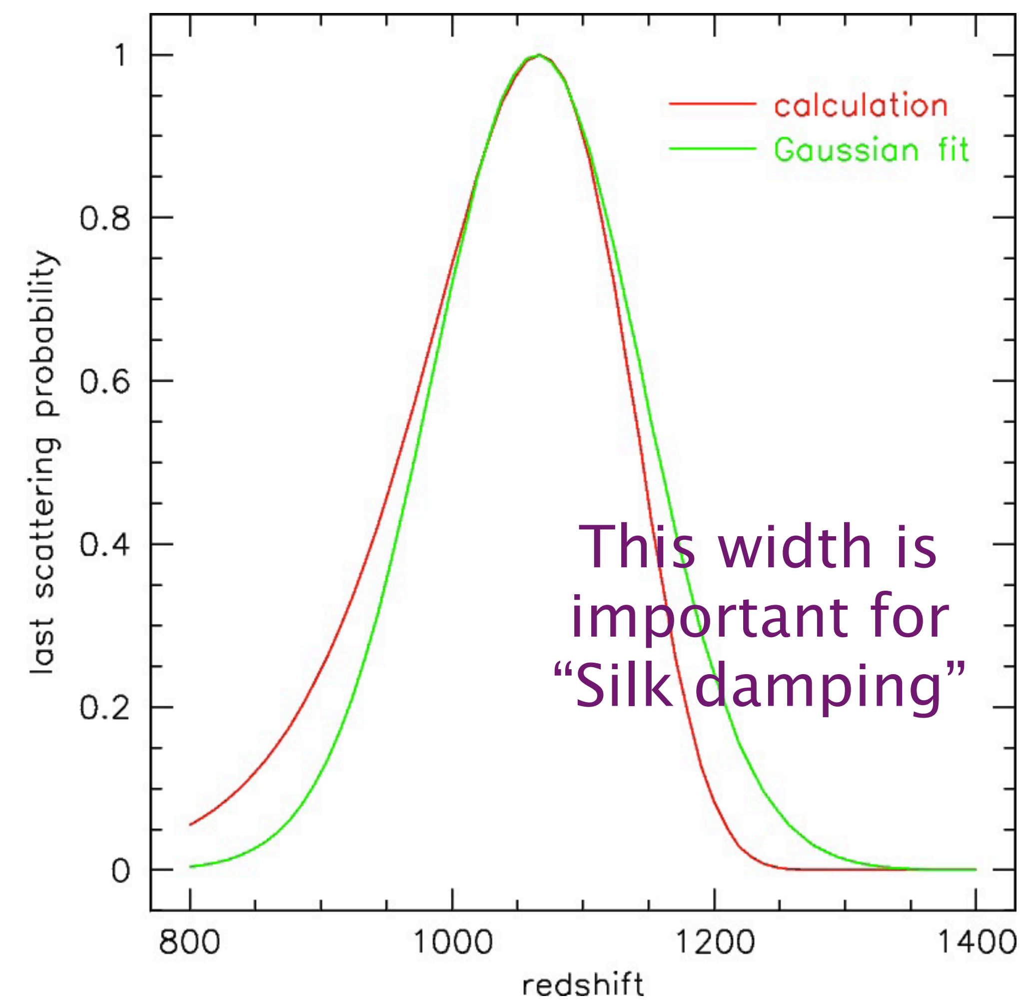


All photons have travelled roughly the same distance since the recombination. We can think of the CMB being emitted from inside of a spherical surface, we’re at the center. This surface has a thickness, just like the surface of a cloud.

The last scattering surface





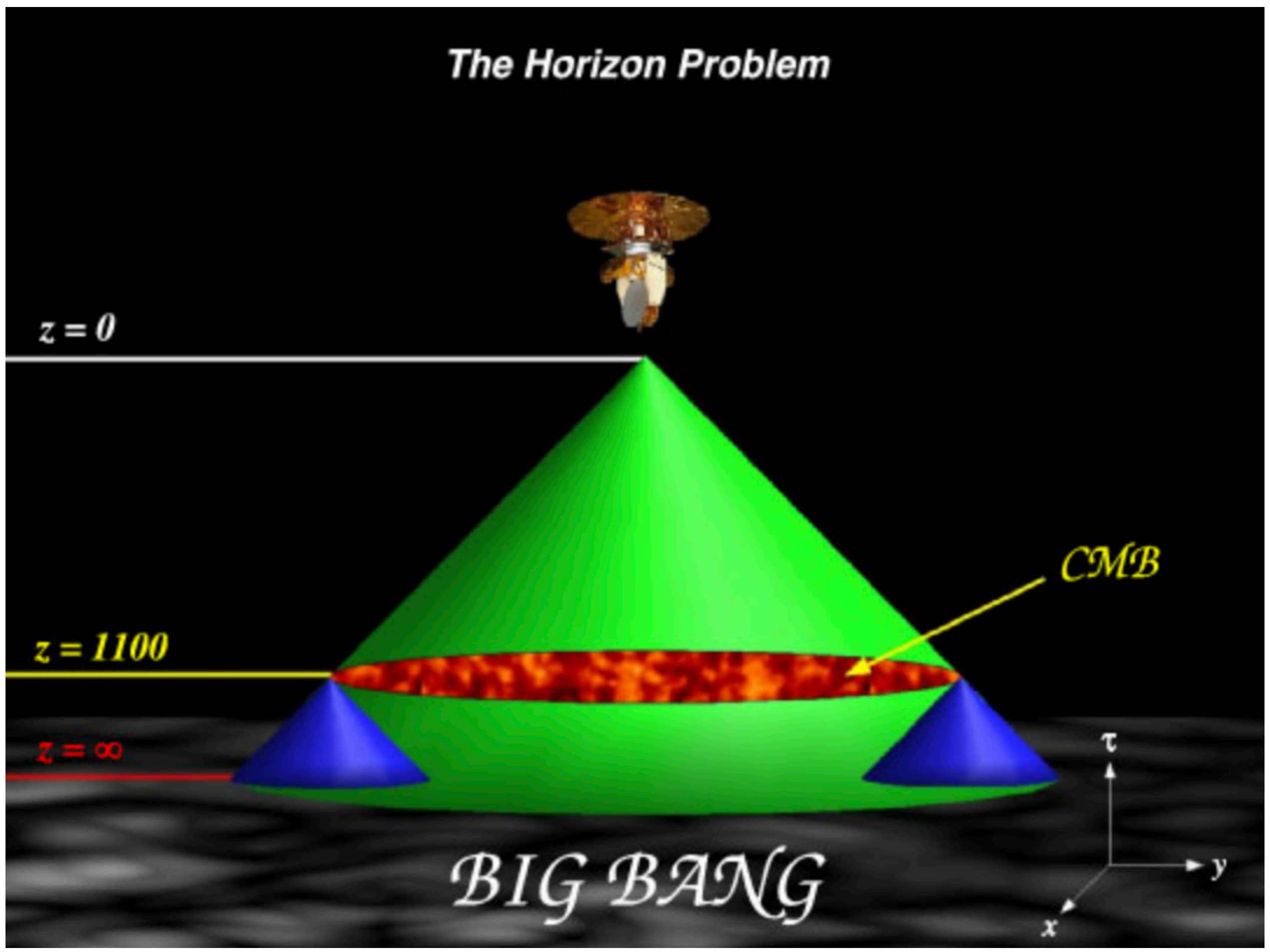


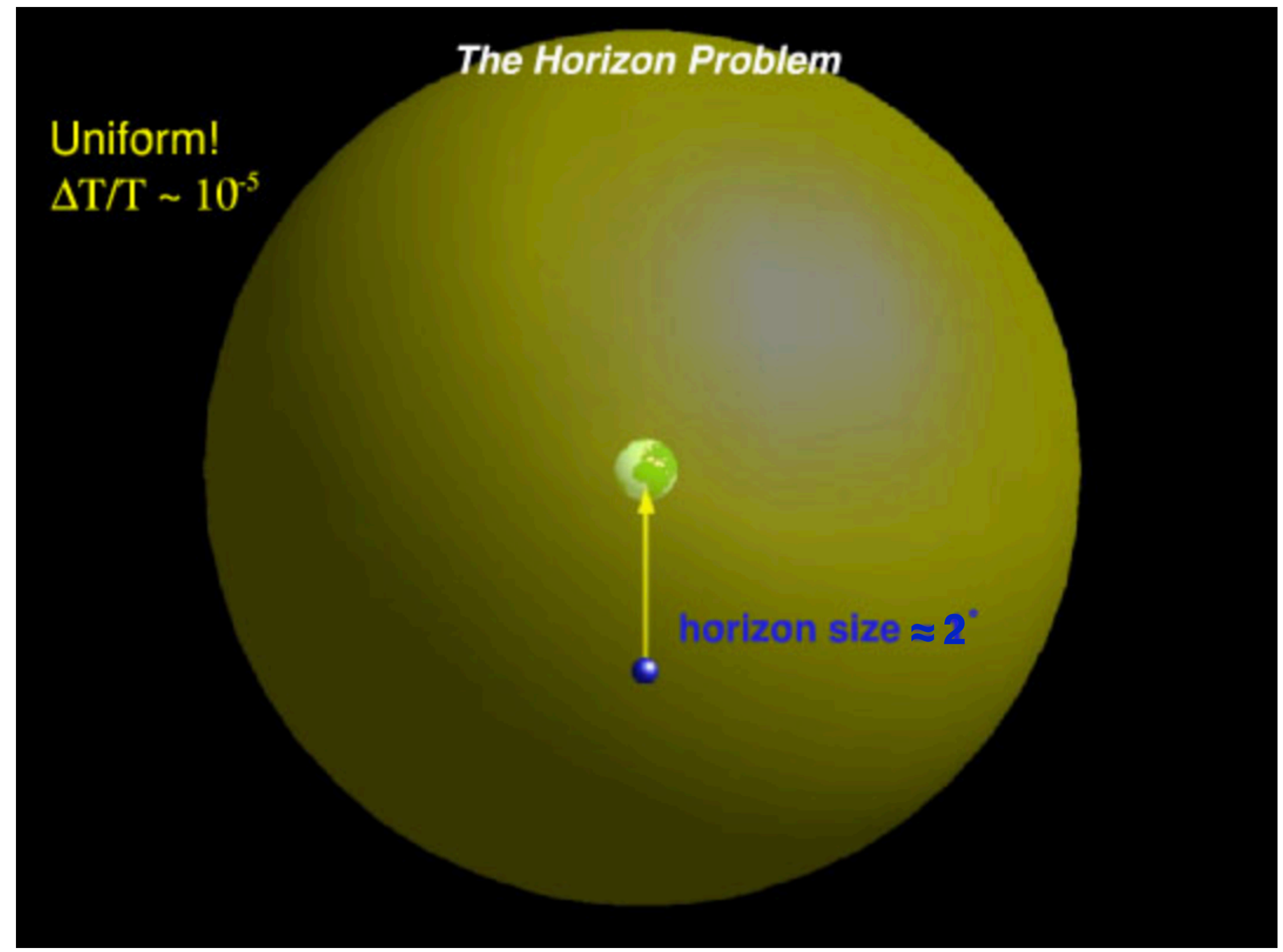
The visibility function is defined as the probability density that a photon is last scattered at redshift z: $g(z) \sim \exp(-\tau)\, d\tau/dz$

Probability distribution is well described by Gaussian with mean $z \sim 1100$ and standard deviation $\delta z \sim 80$.

But why CMB is so uniform?

It might seem natural that the early, hot universe was in thermal equilibrium, but in fact this is quite unnatural! This is because of the cosmological horizon, which is the distance the photons have traveled since the Big Bang, and defines the size of the "sky patch" that could be in *causal contact*.





Simple calculation tells us that scales larger than ~1.7° in the sky were not in causal contact at the time of last scattering. However, the fact that we measure the same mean temperature across the entire sky suggests that even super-horizon scales were once in causal contact. This led to the idea of Inflation.

Cosmology recap

Einstein's field equation reduces to two independent dynamical equations for the scale factor $a(t)$

(see cosmology lecture notes)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}$$

For matter dominated universe (Einstein-de Sitter universe, Ω_m=1), things are particularly simple:

$$\frac{\dot{a}^2}{a^2} \propto a^{-3}, \qquad a(t) \propto t^{2/3}, \qquad H(t) = \frac{2}{3t}, \qquad H(z) = H_0(1+z)^{3/2}$$

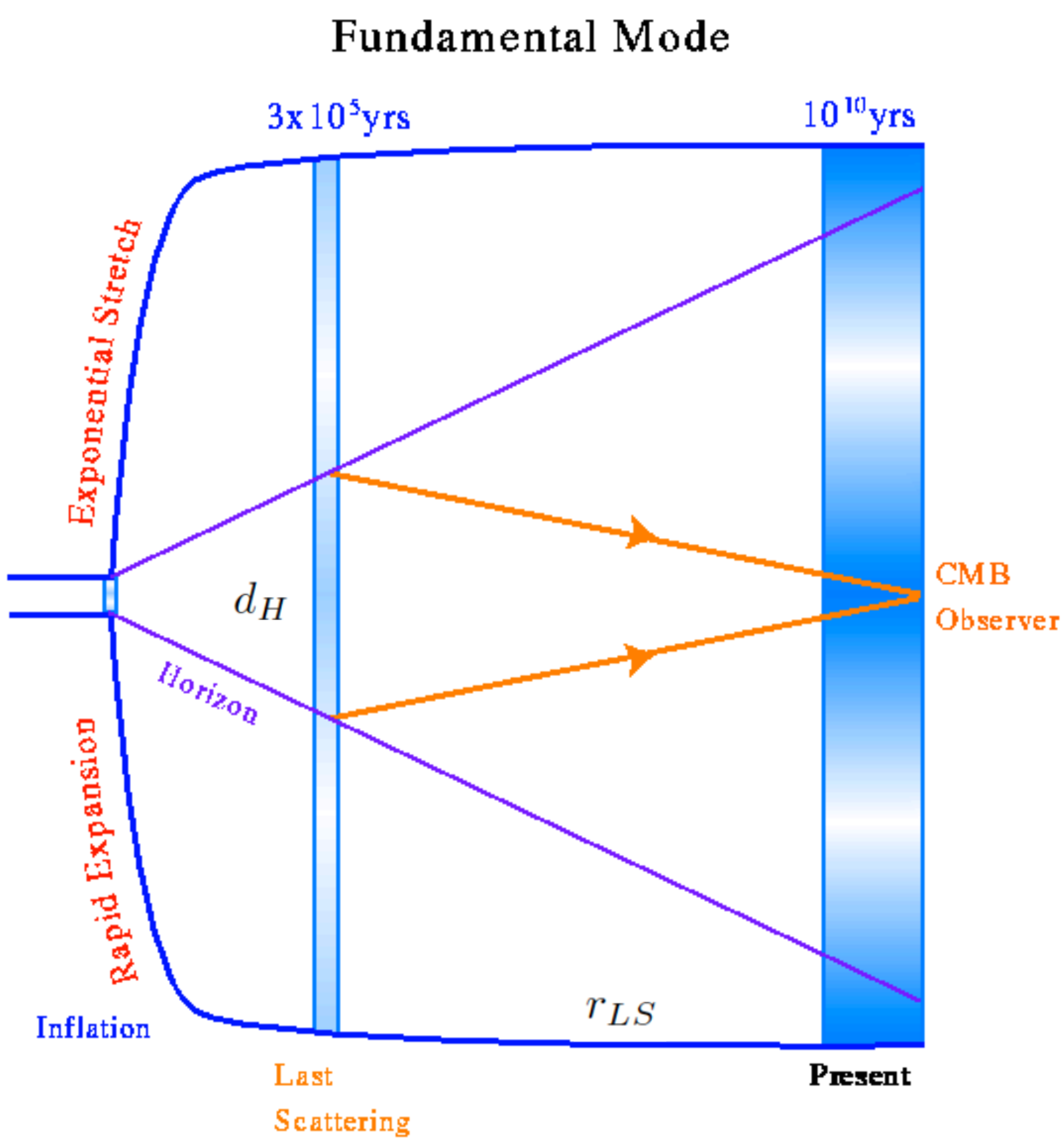
Interesting things happen when the universe is Λ-*dominated* (i.e there's only a cosmological const)!

$$\frac{\dot{a}^2}{a^2} \to \text{constant}, \qquad a(t) \propto \exp(\Lambda t/3), \qquad H = c/R_H = \sqrt{\Lambda/3}.$$

Some definitions. The comoving distance-redshift relation gives us the distance travelled by light in comoving coordinates between two epochs. The conformal time is the time slowed down by the expansion of the universe.

$$\chi = c\int_{z_1}^{z_2} \frac{dz}{H(z)} \qquad\qquad \boxed{\tau = \int \frac{\mathrm{d}t}{a(t)}}$$

Horizon scale at Last Scattering



Distance to the last scattering surface sphere (assume $\Omega_m=1$ EdS universe → $H(z) = H_0(1+z)^{3/2}$)

$$r_{LS} = \frac{c}{H_0}\int_0^{z_{LS}} (1+z)^{-3/2} dz$$
$$= \frac{2c}{H_0}(1-(1+z_{LS})^{-1/2})$$

Thus, the factor $2c/H_0$ is approximately the comoving distance to the LSS ($z_{LS} >> 1$).

The **particle horizon length at the time of last scattering** (i.e. the distance light could travel since big bang) is give by

$$d_H(z=z_{LS}) = \int_{z_{LS}}^{\infty} \frac{c\,dz}{H(z)} = \frac{2c}{H_0}(1+z_{LS})^{-1/2}$$

for $z_{LS} \approx 1100$, means that

$$\theta_H^{LS} = (1+z_{LS})^{-1/2} \approx 1.7^\circ$$

Inflation solves the horizon problem

Inflation turns the horizon problem on its head! By **postulating a phase of the universe with accelerated expansion**, it can be shown that the comoving scales grow more quickly than the horizon, and the universe evolves towards flatness rather than away. Two points that were in causal contact initially ($d < d_H$) will expand so rapidly that they will be causally *disconnected*.

The density evolution of the universe, filled with "matter" with equation of state $\boldsymbol{p = w\rho}$ (in c=1 units), is shown on the right. For ordinary matter (w=0) or radiation (w=1/3) we get decelerated expansion and an increasing Hubble radius.

$$\boxed{\frac{d\rho}{dt} + 3H(\rho + p) = 0} \Rightarrow \frac{d\ln\rho}{d\ln a} = -3(1+w)$$

$$w \equiv \frac{p}{\rho}$$

comoving Hubble radius, $(aH)^{-1}$

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)} \qquad \tau \propto a^{\frac{1}{2}(1+3w)}$$

From the equation for the acceleration of the universe, we immediately see that the condition for acceleration ($\ddot{a} > 0$) is that we have negative pressure, i.e. $1+3w < 0$. In this phase the Hubble radius shrinks with time.

$$\frac{\ddot{a}}{a} = -(1+3w)\left(\frac{4\pi G}{3}\rho\right) \qquad \boxed{\frac{d}{dt}\left(\frac{1}{aH}\right) < 0}$$

$$\boxed{\frac{d}{dt}\left(\frac{H^{-1}}{a}\right) < 0 \quad \Rightarrow \quad \frac{d^2a}{dt^2} > 0 \quad \Rightarrow \quad \rho + 3p < 0}$$

In an inflationary universe, there is no singularity a=0 at the conformal time τ=0. During the exponentially expanding phase, conformal time is negative and evolves towards zero.

$$a(t) \propto e^{Ht}, \;\; H = \text{const.} \qquad \tau = \int \frac{dt}{a(t)} \propto -\frac{1}{H}e^{-Ht}, \qquad a(\tau) = -\frac{1}{H\tau}.$$

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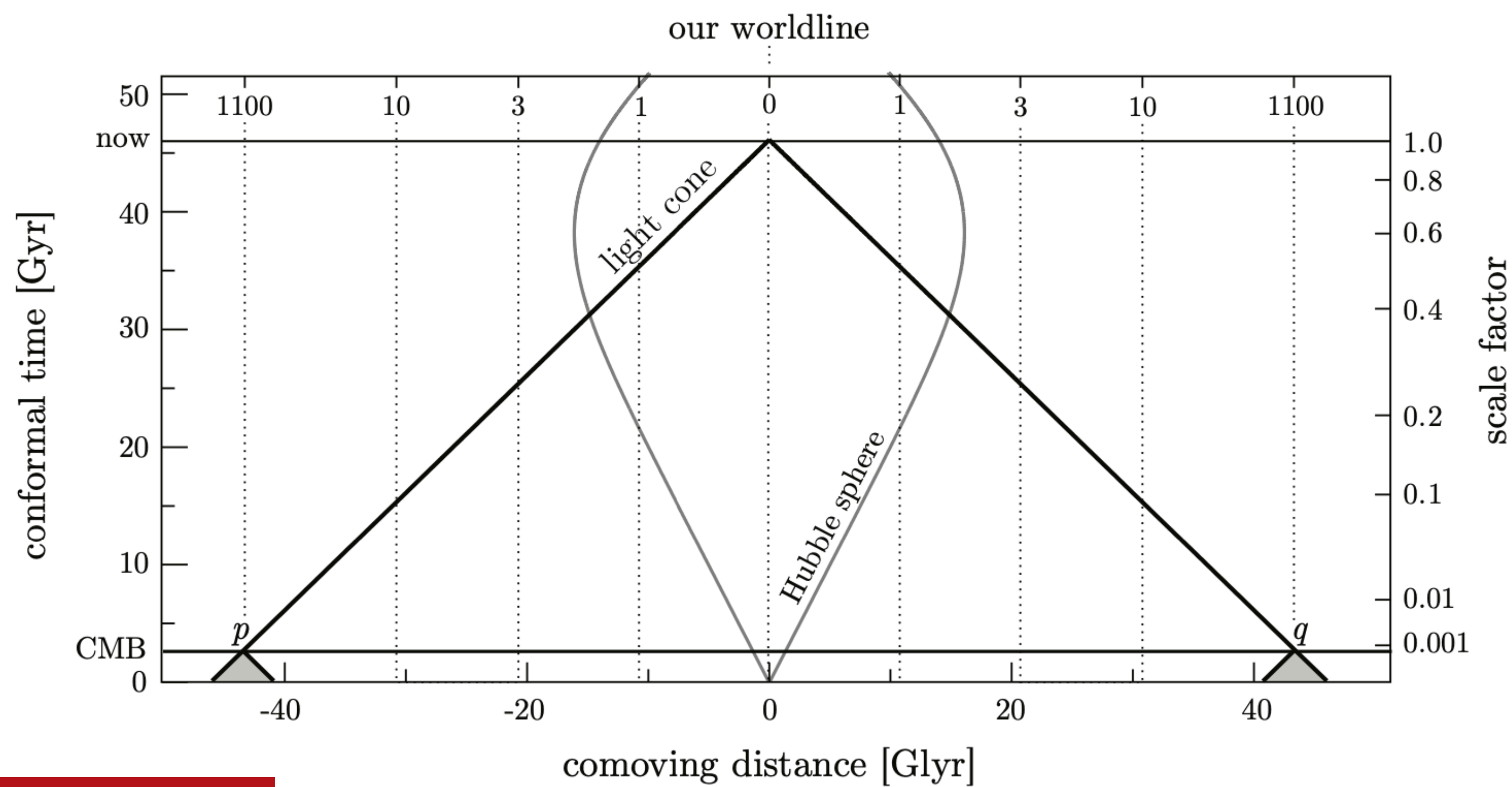
Horizon problem in the conventional Big Bang model. Our past light cone intersects the space-like slice marked CMB (the last scattering surface) at points p and q, whose own light cones don't overlap, as they hit the singularity at τ=0.



arXiv > hep-th > arXiv:0907.5424

High Energy Physics – Theory

[Submitted on 30 Jul 2009 (v1), last revised 30 Nov 2012 (this version, v2)]

TASI Lectures on Inflation

Daniel Baumann

arXiv.org > astro-ph > arXiv:astro-ph/0301448

Astrophysics

[Submitted on 22 Jan 2003 (v1), last revised 21 Jul 2004 (this version, v2)]

Cosmology, inflation, and the physics of nothing

William H. Kinney (ISCAP, Columbia Univ.)

Image credit: Daniel Baumann

(see www.damtp.cam.ac.uk/user/db275/Cosmology/Lectures.pdf)

Inflation solves the horizon problem

Inflation turns the horizon problem on its head! By **postulating a phase of the universe with accelerated expansion**, it can be shown that the comoving scales grow more quickly than the horizon, and the universe evolves towards flatness rather than away. Two points that were in causal contact initially *($d < d_H$)* will expand so rapidly that they will be causally *disconnected*.

Inflationary scenario solves the horizon problem! The Hubble sphere shrinks during inflation and the conformal time is negative. The space-like singularity at τ=0 is replaced by the reheating (end-of-inflation) surface.

This is τ=0, but the Big Bang singularity has now been replaced by the "reheating surface" at the end of inflation

The universe goes through roughly 60 e-folding during this inflationary expansion (this number is model-dependent). All points on the CMB line has overlapping light cones and come from causally connected regions of space.

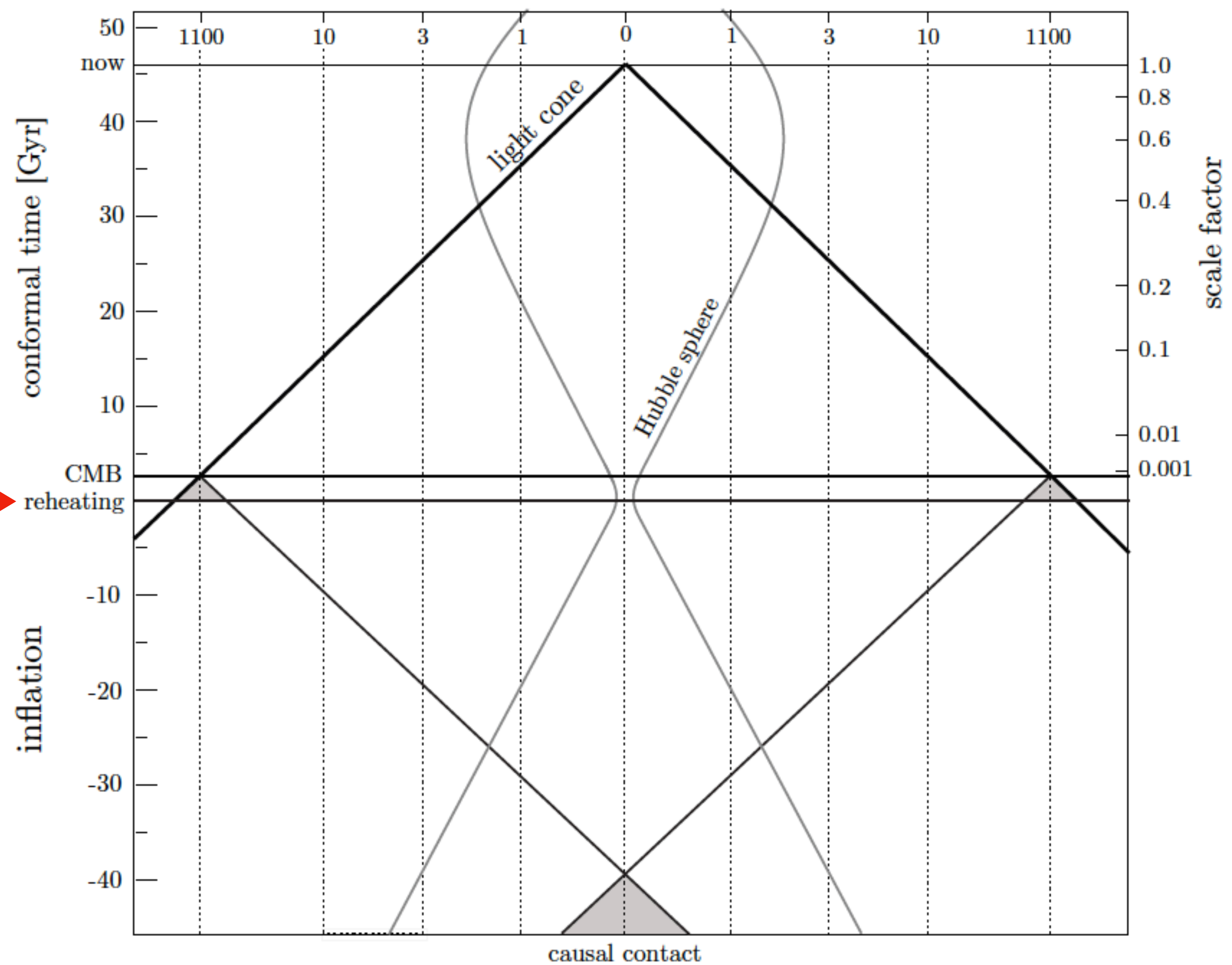


Image credit: Daniel Baumann

(see www.damtp.cam.ac.uk/user/db275/Cosmology/Lectures.pdf)

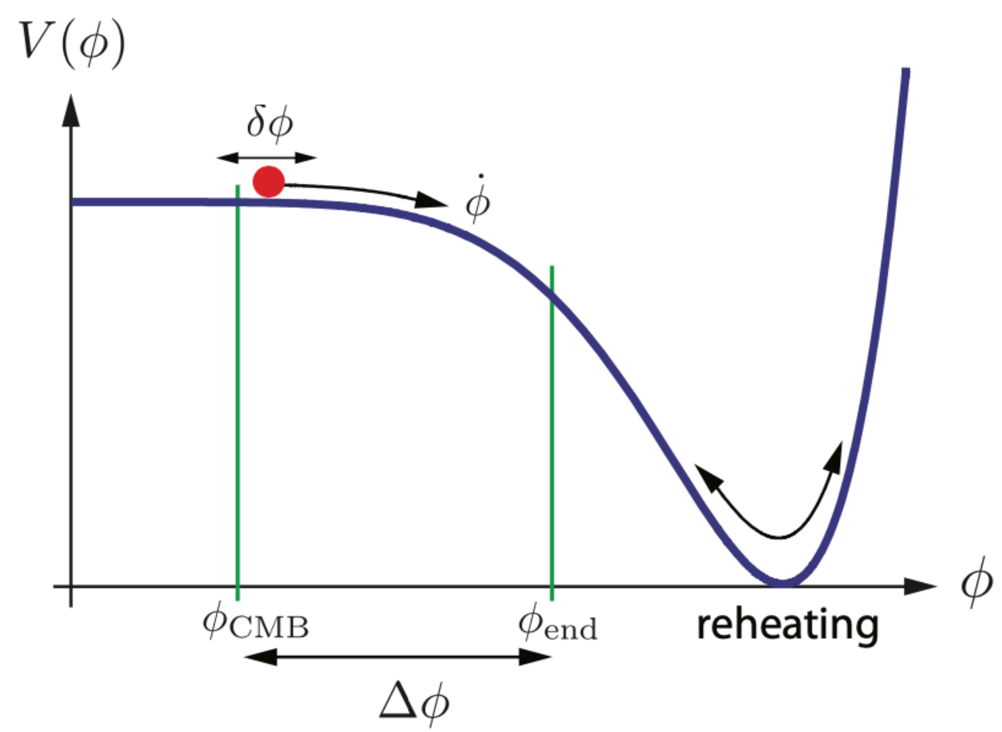
Mechanism of inflation

Inflation occurs if the universe is filled with a scalar field φ, which has non-vanishing scalar potential $V(\varphi)$ (it cannot be constant because inflation must end!). This homogeneous field φ then satisfies the equation

$$\ddot{\varphi} + 3H\dot{\varphi} = -\frac{dV}{d\varphi}. \qquad a(t) \propto \exp\left(\int H dt\right), \quad H \approx \text{const.}$$

For a relatively flat potential ($dV/d\phi$ small), the acceleration term can be neglected. The Friedmann equation in this case is $H^2 = 8\pi/3G\, V(\varphi)$. If φ varies slowly, then $V(\varphi)$ and thus H also varies slowly, and the parameters of inflation are almost time independent. This is known as a *slow-roll inflation*.

More on inflation during the CMB polarization lectures



Slow-roll parameters, which determines the number of e-foldings, are directly measurable from CMB experiments.

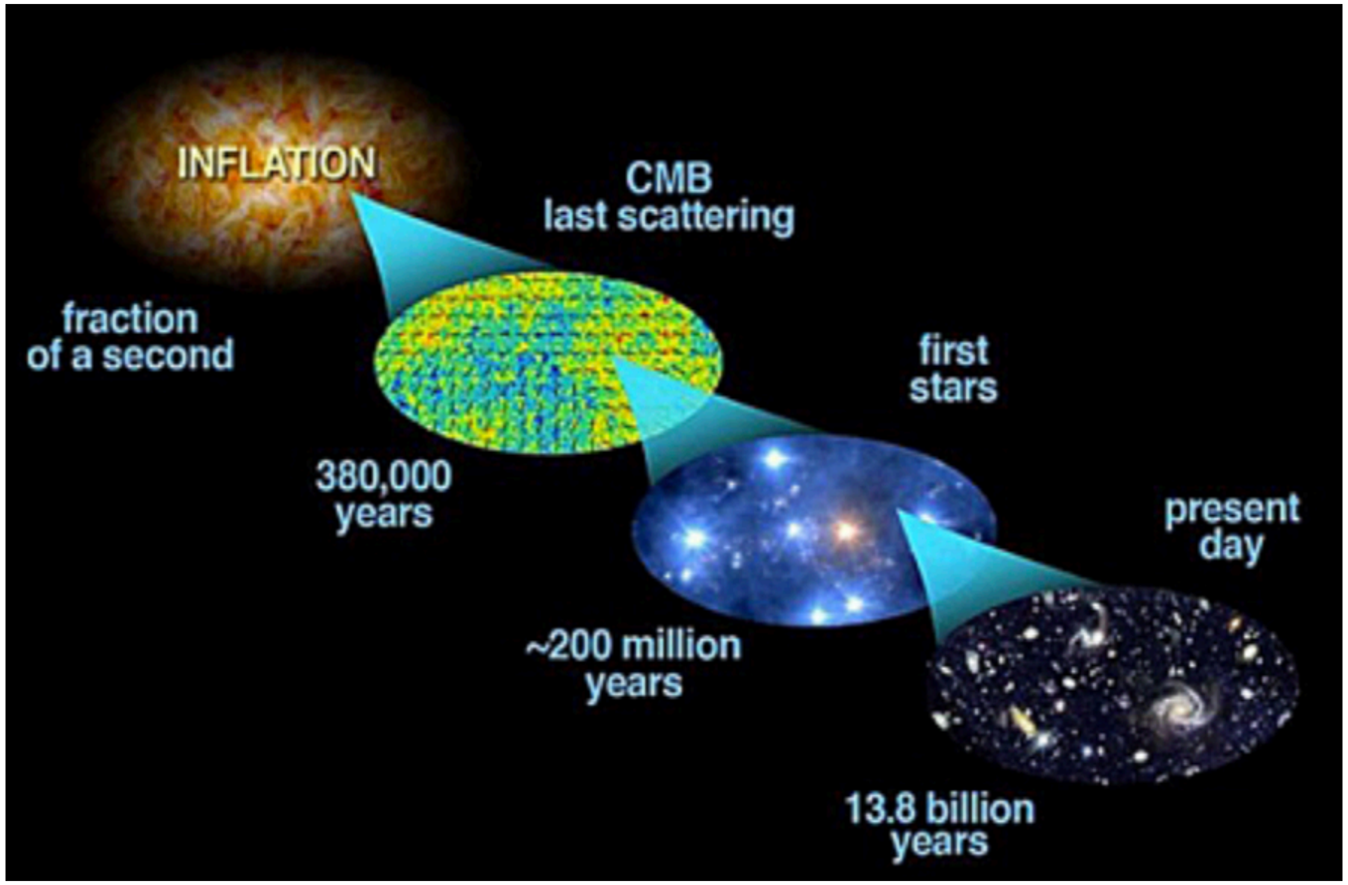
$$H^2 = \frac{V(\varphi)}{3M_{\rm Pl}^2} \qquad 3H\dot{\varphi} = -V'(\varphi)$$

$$\varepsilon(\varphi) \equiv \frac{1}{2}M_{\rm Pl}^2\left(\frac{V'}{V}\right)^2 \qquad \eta(\varphi) \equiv M_{\rm Pl}^2\frac{V''}{V}$$

Example of an inflaton potential. Acceleration occurs when the potential energy of the field, $V(\phi)$, dominates over its kinetic energy, $\frac{1}{2}\dot{\phi}^2$. Inflation ends at $\phi_{\rm end}$ when the kinetic energy has grown to become comparable to the potential energy, $\frac{1}{2}\dot{\phi}^2 \approx V$. CMB fluctuations are created by quantum fluctuations $\delta\phi$ about 60 e-folds before the end of inflation. At reheating, the energy density of the inflaton is converted into radiation.

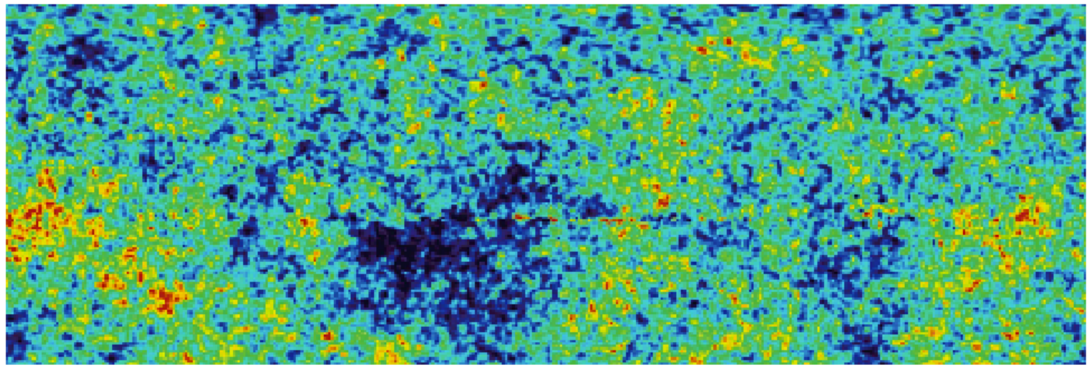
Inflation and the CMB

Inflationary scenario also solves the **flatness problem, origin of fluctuations problem, magnetic monopole (and other superheavy relics) problem**. And the main support for inflation comes from the CMB.



Observation of the CMB provides evidence of a remarkable story: That we all came from quantum fluctuations!

- Fluctuations we observe today in CMB and the matter distribution originate from quantum fluctuations during inflation



Mukhanov&Chibisov (1981)
Guth & Pi (1982)
Hawking (1982)
Starobinsky (1982)
Bardeen, Steinhardt&Turner (1983)

Predictions of inflation & the CMB

Inflationary models make **specific set of predictions that can be verified with CMB data:**

- Small spacial curvature
- Nearly scale-invariant spectrum of density perturbations
- CMB temperature anisotropies from large to small angular scales (Sachs-Wolfe effect and acoustic peaks)
- Gaussian perturbations
- Existence of primordial gravity waves!

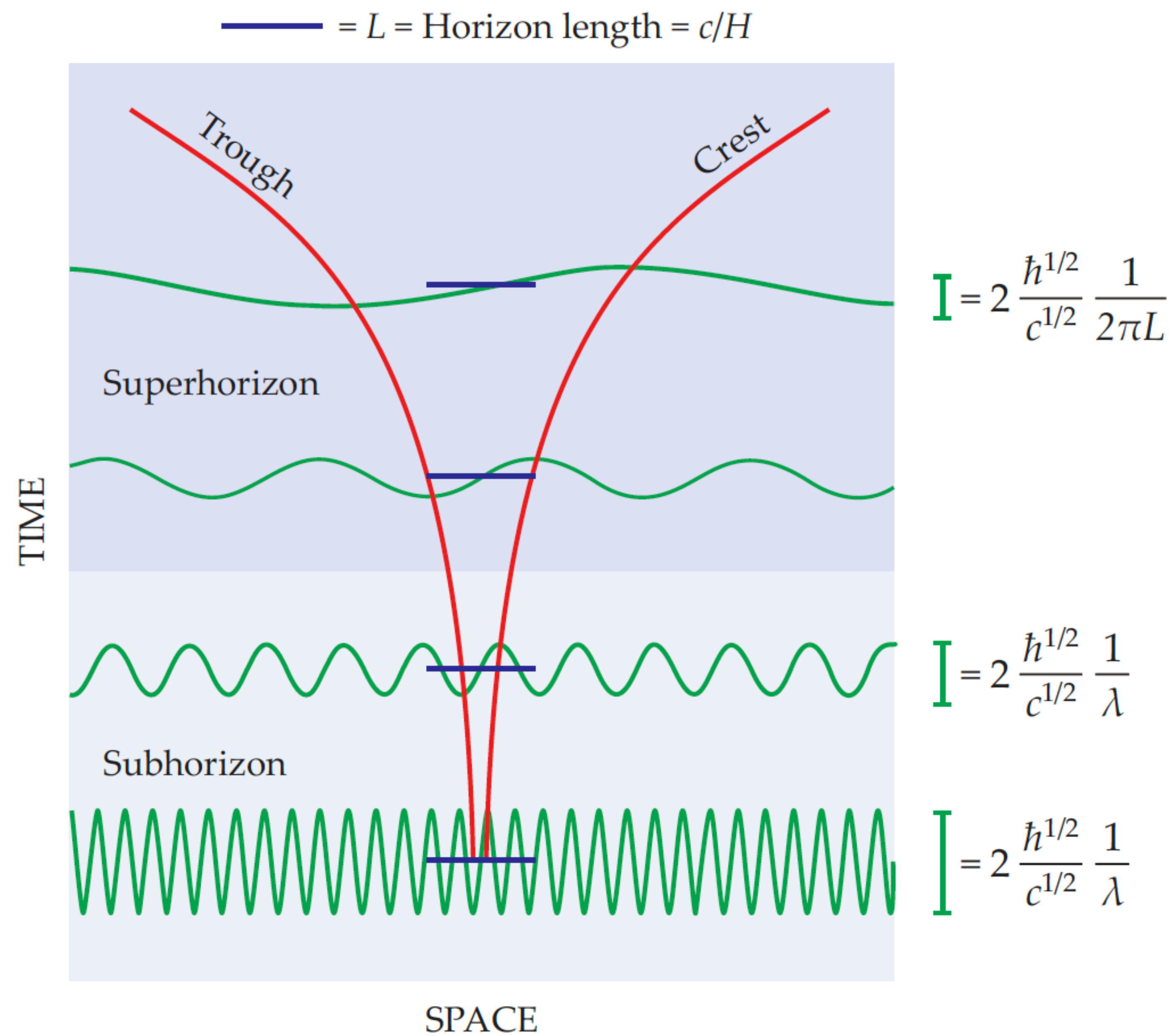
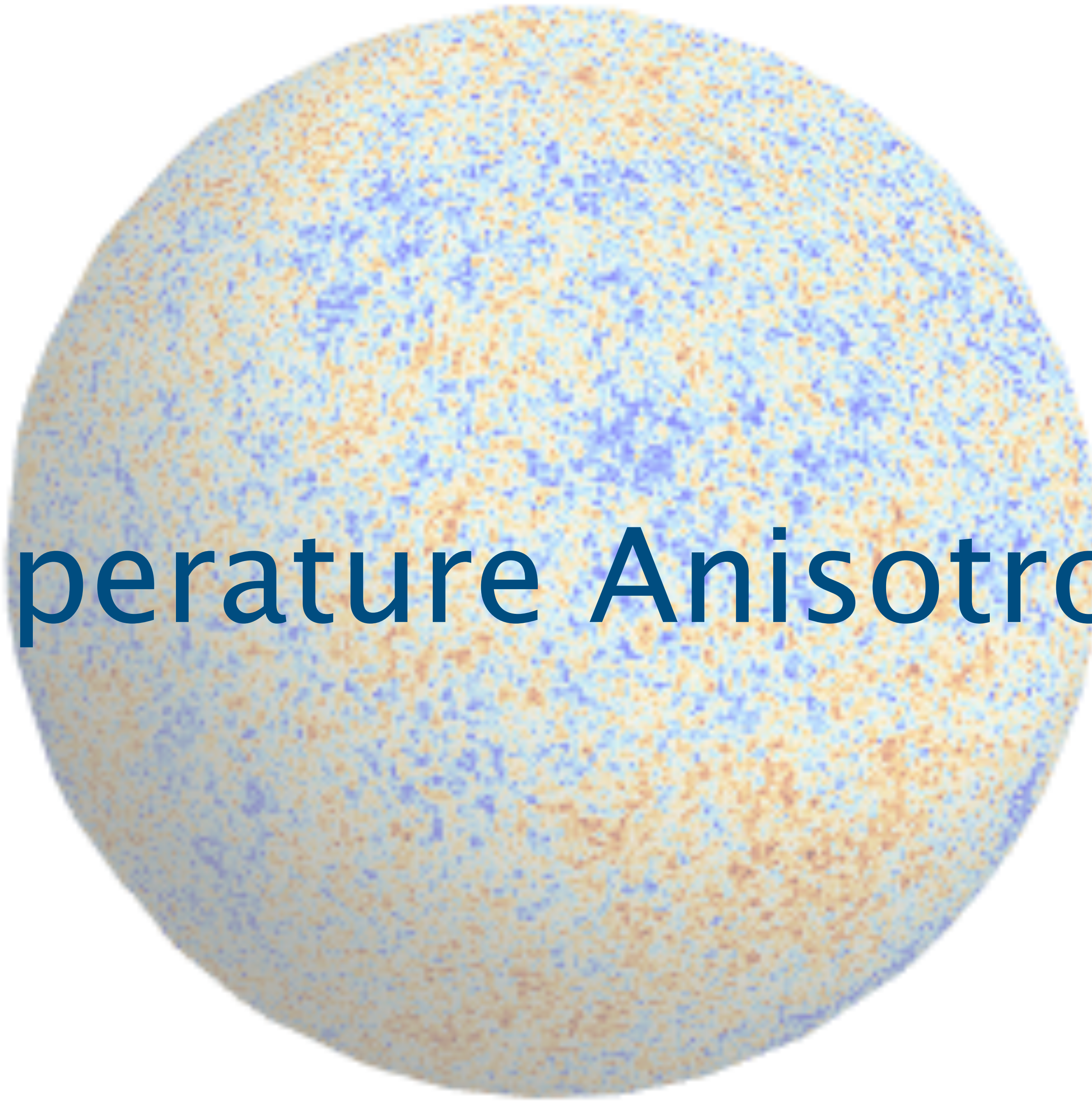
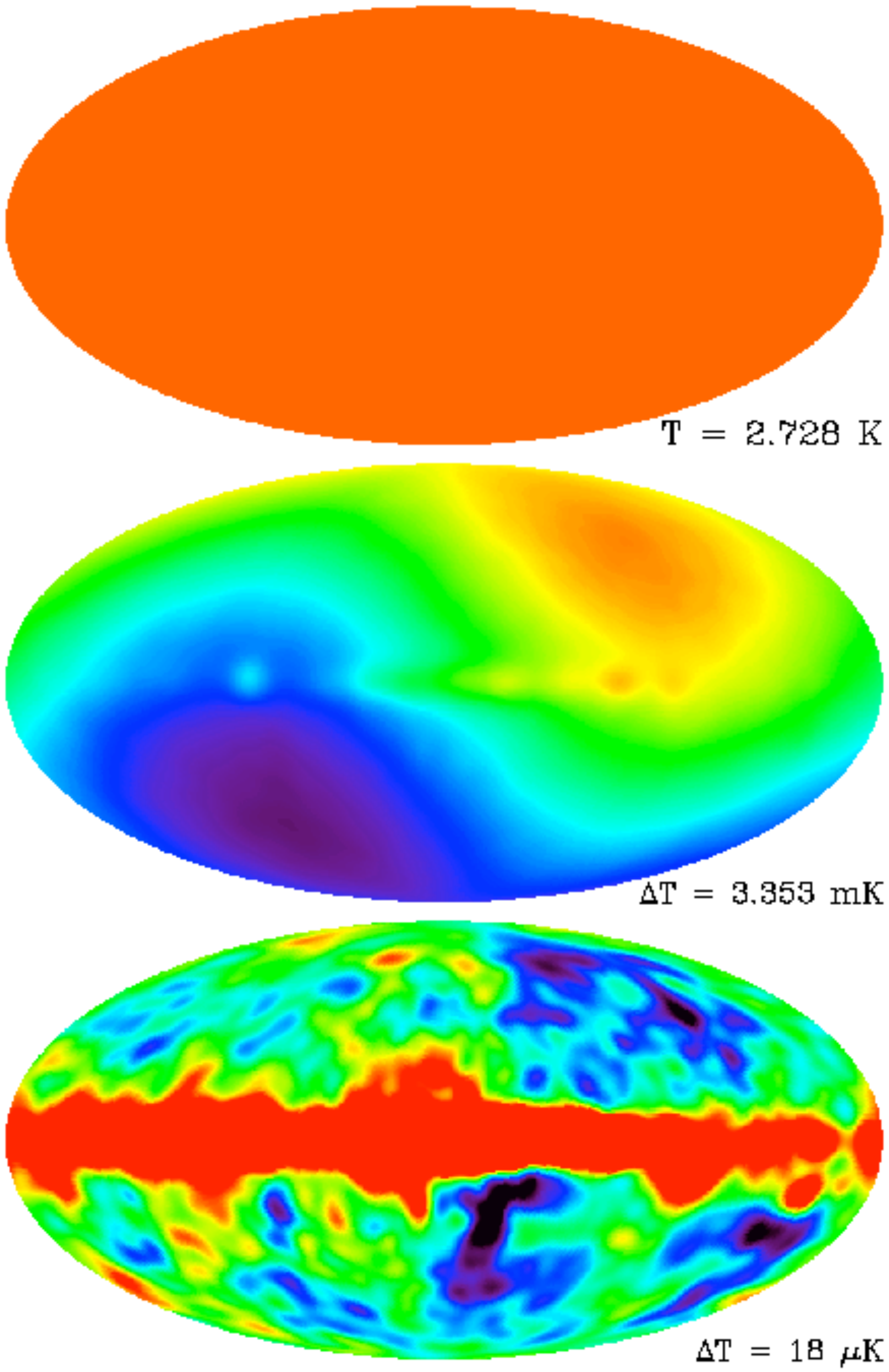


Figure from Carlstrom, Crawford & Knox (2015), Physics Today



Temperature Anisotropies

Amplitude of the temperature anisotropies



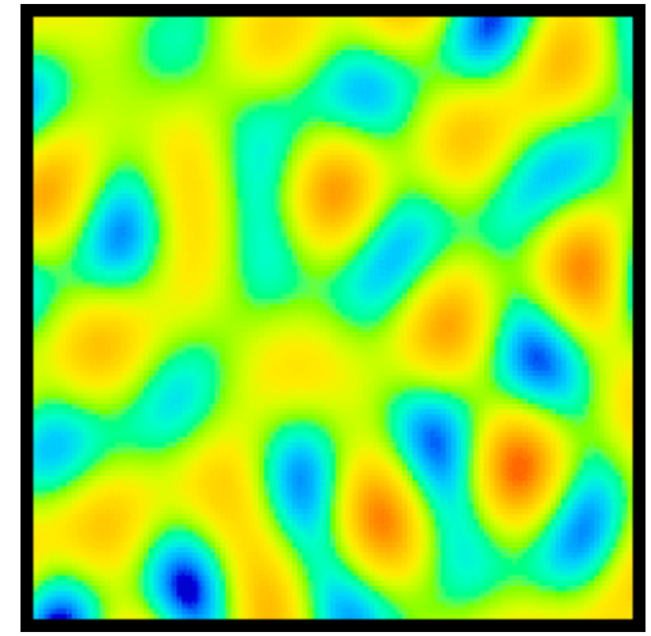
CMB is primarily a uniform glow across the sky. But..

Turning up the contrast, a dipole pattern becomes prominent at a level of 10^{-3}. This is from the motion of the Sun relative to the CMB monopole (rest frame).

Enhancing the contrast further (at the level of 10^{-5}, and after subtracting the dipole, temperature anisotropies appear.

Temp. anisotropies & dark matter

From the fact that non-linear structures exist today in the Universe, the linear growth theory predicts that density perturbations at z = 1100 (the time of CMB release) must have been of the order of



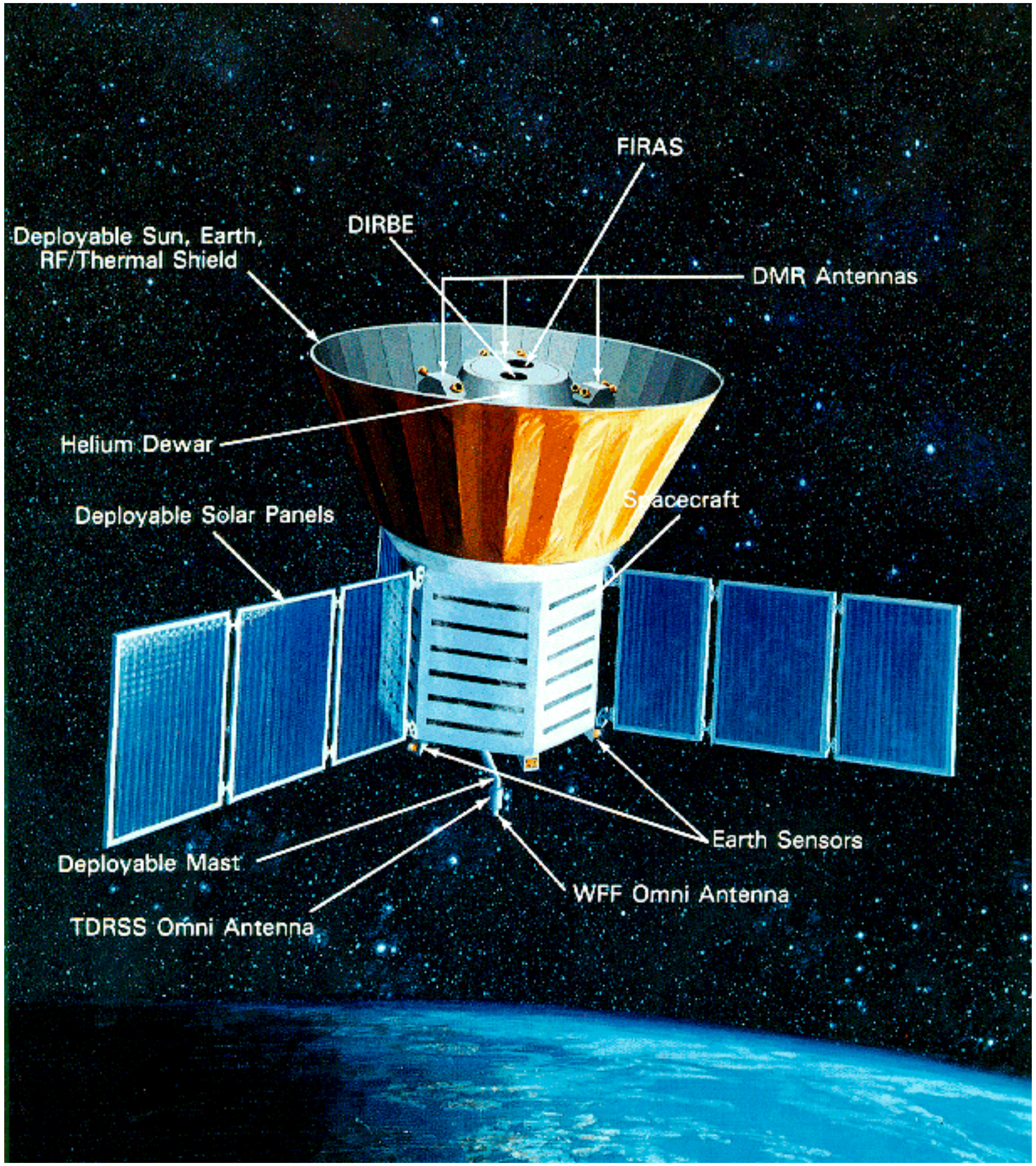
$$\delta(a_{\mathrm{CMB}}) = \frac{\delta(a=1)}{D_+(a_{\mathrm{CMB}})} \gtrsim 10^{-3}$$

After the CMB was found in 1965, fluctuations were sought at the relative level of 10^{-3}, but they were not found. Eventually they were found at a level of 10^{-5}.

The reason is that density contrast we see *today* is dominated by dark matter, while the CMB temperature differences are couple to baryons. DM perturbations grow independently of the baryons. While the radiation-baryon fluid oscillated and therefore couldn't grow in amplitude, the DM perturbations continued to grow. Since DM has no coupling to the electromagnetic spectrum nor to the baryons, this growth happened without pumping the perturbations in the CMB to equal levels.

In fact, this can be seen as a proof that a *baryonic universe* can not form the present structures, and a non-interacting form of matter must dominate!

COBE satellite



Credit: NASA

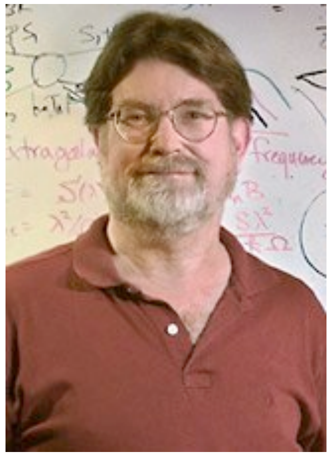
Launched on Nov. 1989 on a Delta rocket.

DIRBE: Measured the absolute sky brightness in the 1–240 μm wavelength range, to search for the Infrared Background

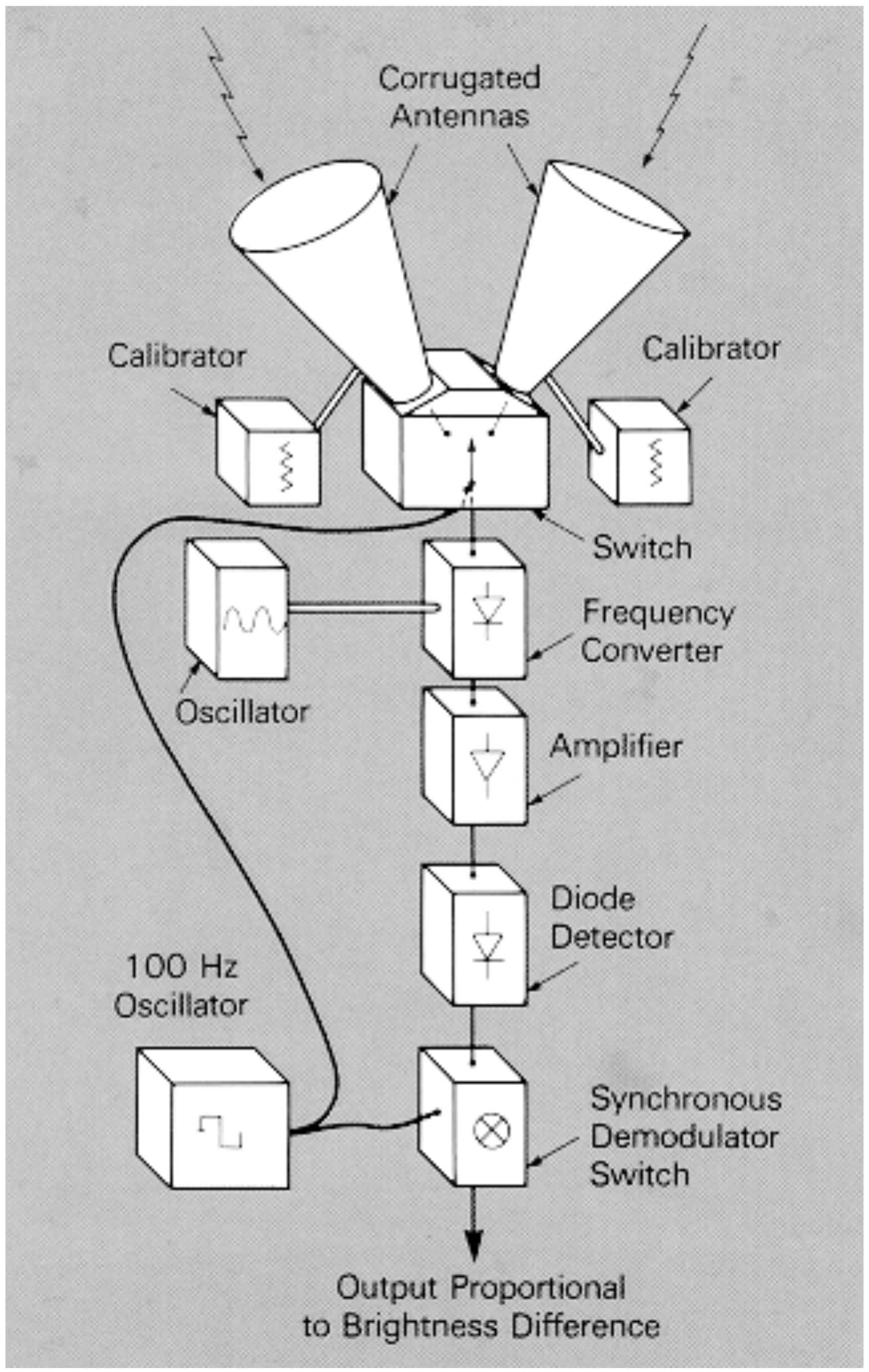
FIRAS: Measured the spectrum of the CMB, finding it to be an almost perfect blackbody with $T_0 = 2.725 \pm 0.002$ K

DMR: Found “anisotropies” in the CMB for the first time, at a level of 1 part in 10^5

2006 Nobel prize in physics

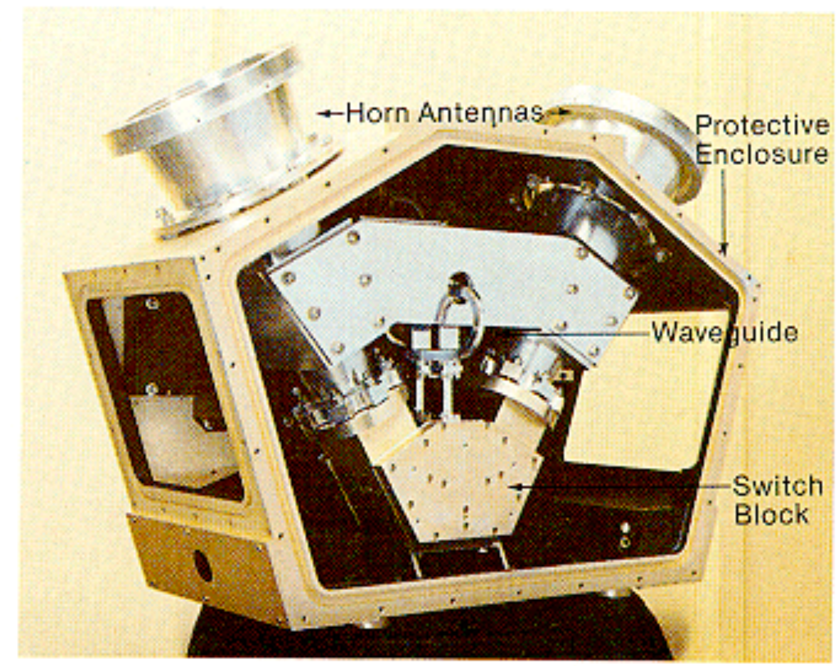


DMR on COBE



Credit: NASA

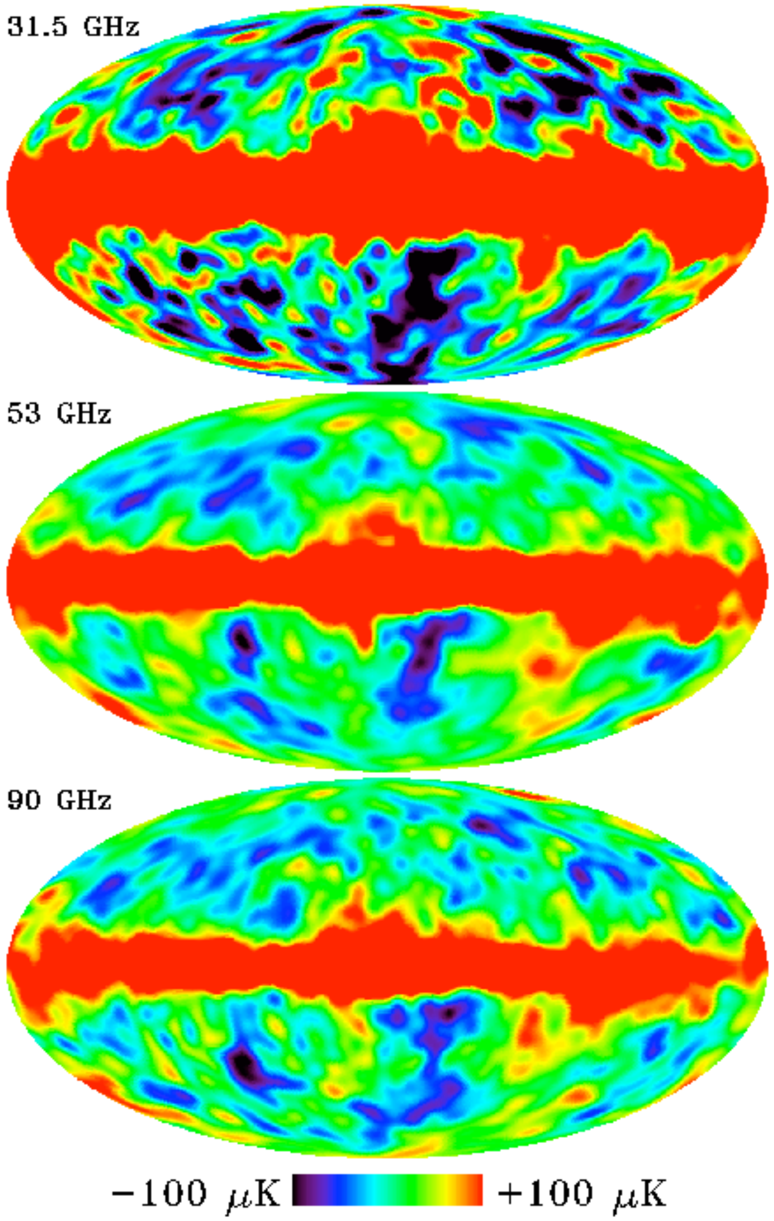
Differential Microwave Radiometer

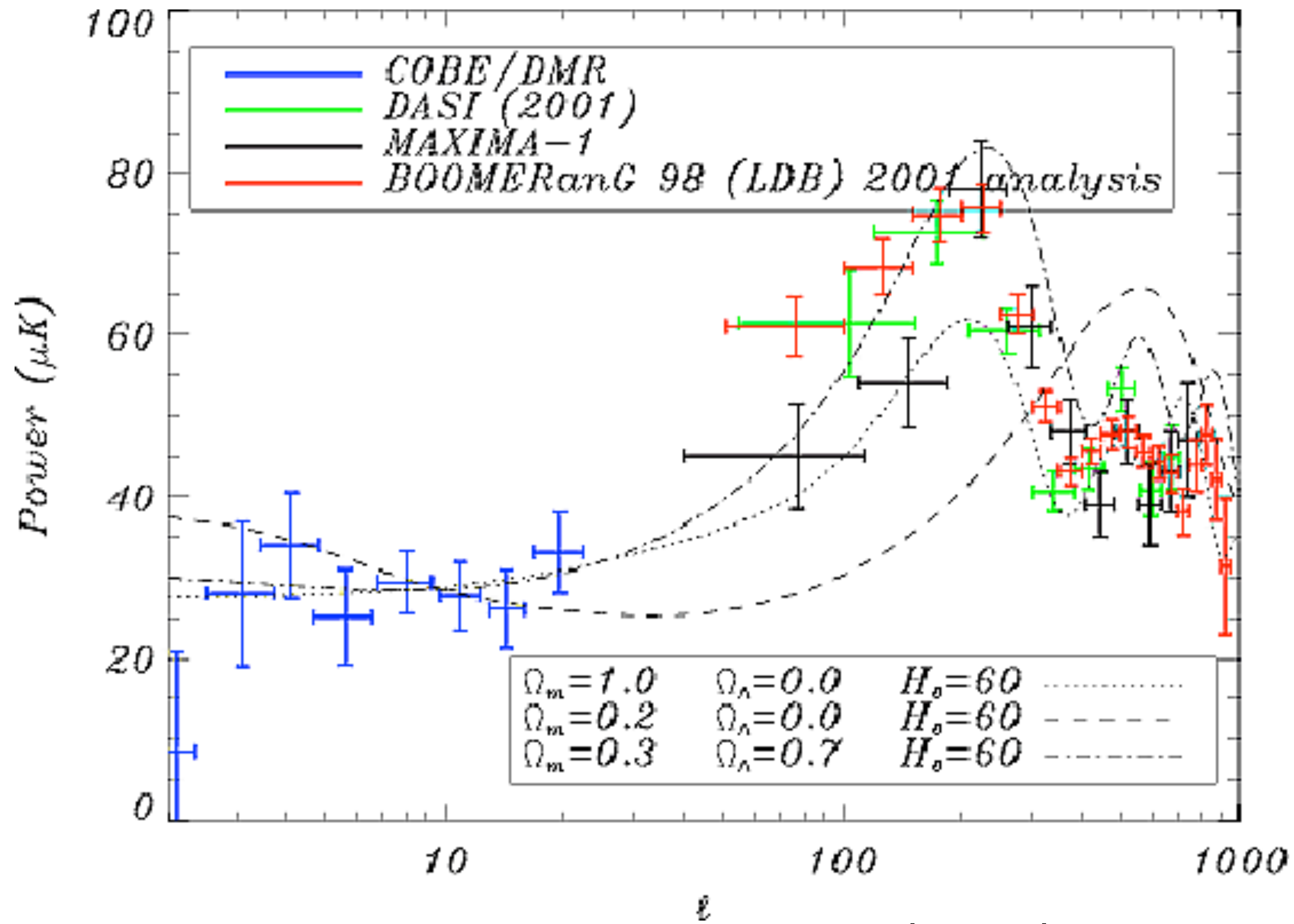


The 9.6 mm DMR receiver partially assembled. Corrugated cones are antennas.

- Differential radiometers measured at frequencies 31.5, 53 and 90 GHz, over a 4-year period
- Comparative measurements of the sky offer far greater sensitivity than absolute measurements

COBE DMR Measurements



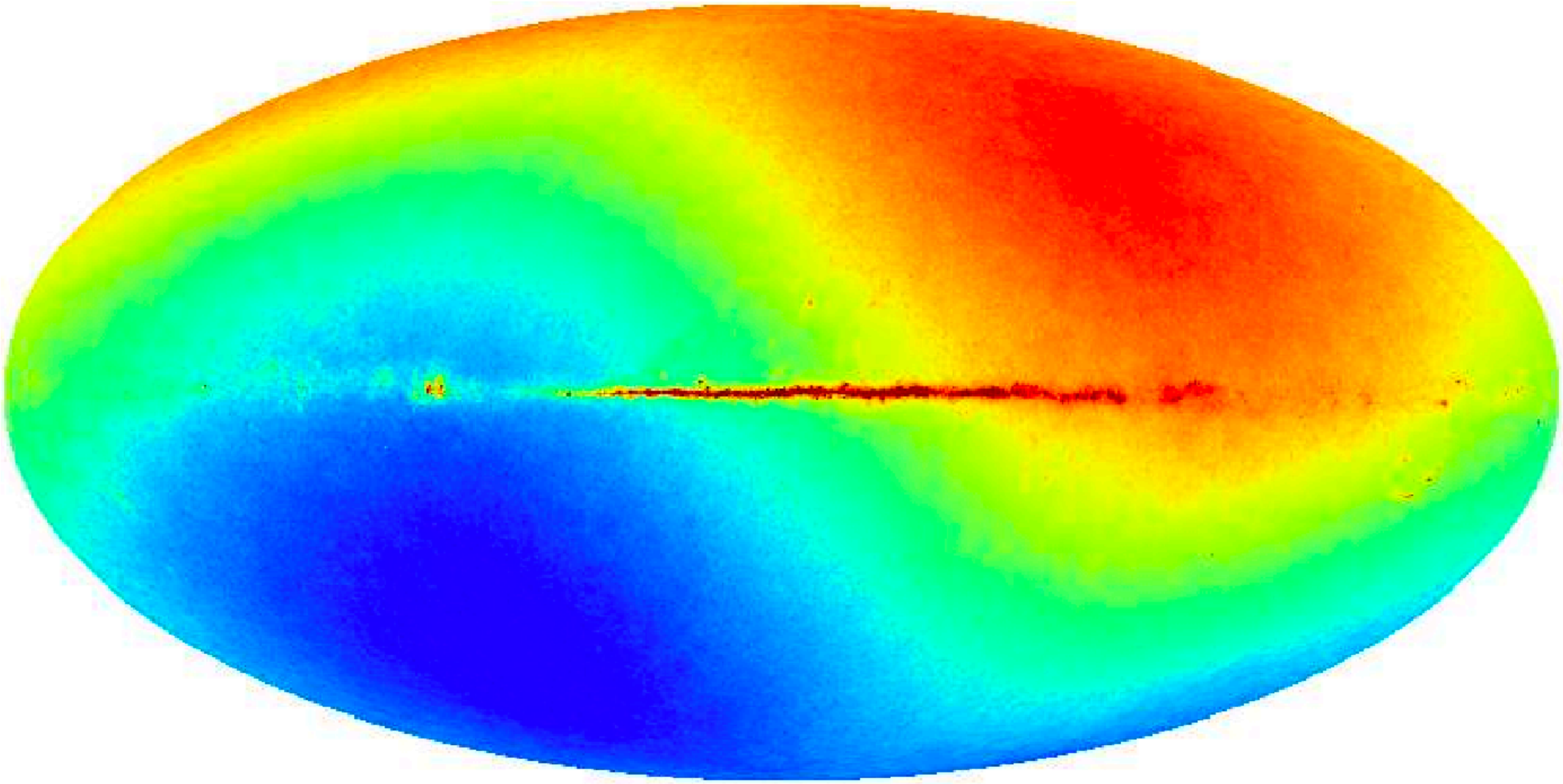


Credit: Archeops team

COBE DMR results
First announced in Smoot et al. (1992)

2006 Nobel Prize in Physics for George Smoot

The CMB dipole



The actual 100 GHz channel map from the *Planck* satellite, showing the dipole variation of intensity across the sky. The corresponding temperature difference is 3.36 mK.

- Measured velocity: 390±30 km/s
- After subtracting out the rotation and revolution of the Earth, the velocity of the Sun in the Galaxy and the motion of the Milky Way in the Local Group one finds: v = 627 ± 22 km/s
- Towards Hydra-Centaurus, l=276±3° b=30±3°

Can we measure an intrinsic CMB dipole ?

Velocity induced change on CMB

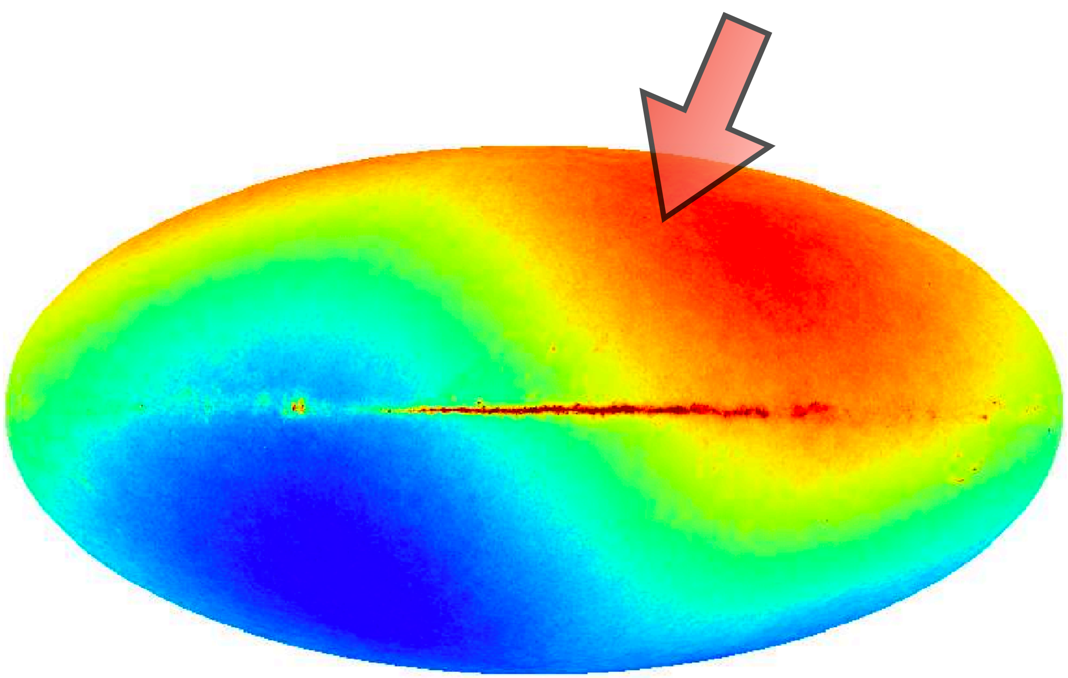
If you are moving with velocity β with respect to the CMB blackbody, then the intensity change in the direction μ=cos(θ) would be

$$I_\nu = I_0\ x^3 \left[\exp\left(x \frac{1+\beta\mu}{\sqrt{1-\beta^2}}\right) - 1\right]^{-1}$$

This follows from the simple formula for relativistic Doppler boost: $\frac{\nu}{\nu'} = \frac{1+\beta\mu}{\sqrt{1-\beta^2}}$

Expanding in powers of β, we thus get

$$\frac{I_\nu}{I_0} = \frac{x^3}{e^x - 1}\left[1 - \frac{xe^x}{e^x-1}\mu\beta + \frac{xe^x}{e^x-1}\left(\left(\frac{x}{2}\right)\frac{e^{x/2}+1}{e^{x/2}-1}\mu^2 + \frac{1}{2}\right)\beta^2 + \mathcal{O}(\beta^3)\right]$$



The second term is just the well-known CMB dipole, with frequency dependence of a pure temperature change. The second order term contains a y-distortion, from the mixing of blackbodies, and also a contribution to the pure temperature dipole.

Velocity induced change on CMB

Now, if there is also temperature fluctuations (as indeed we have!), then there will also be aberration and modulation of those fluctuations. To see that, we write the anisotropic CMB sky intensity in a general way:

$$\frac{\Delta I(\hat{\mathbf{n}})}{I} = f(x)\left[\frac{\Delta T(\hat{\mathbf{n}})}{T} + y(\hat{\mathbf{n}})Y(x)\right]$$

where $f(x) = xe^x/(e^x - 1)$ comes from the derivative of a blackbody at a fixed temperature, and $Y(x) = x[\frac{e^x+1}{e^x-1} - 4]$ is the y-distortion spectrum.

Again, applying a velocity boost to this and expanding in powers of β, we get

aberration modulation

$$\frac{\Delta I'(\hat{\mathbf{n}}')}{I_\nu f(x)} = \beta\mu + \frac{\Delta T'(\hat{\mathbf{n}}')}{T_{\rm CMB}}(1+3\beta\mu) + Y(x)\left[y(\hat{\mathbf{n}}')(1+3\beta\mu) + \beta\mu\frac{\Delta T(\hat{\mathbf{n}}')}{T_{\rm CMB}}\right] + \beta\mu\, y\left[Y^2(x) - x\frac{\mathrm{d}Y(x)}{\mathrm{d}x}\right] + \mathcal{O}(\beta^2)$$

The first term is again the temperature dipole, showing a change of intensity. The second term gives the modulation and aberration effects, both linear in β (so about 1000 times smaller than the intensity dipole). The first term in the second line gives modulation and aberration of the y-distortion anisotropies, whereas the second term is an additional y-modulation from temperature anisotropies. (Note that these effects are not unique to CMB, they occur for all astronomical observations!)

Direction of incoming photons changes from $\hat{\mathbf{n}}$ to $\hat{\mathbf{n}}'$

$$\hat{\mathbf{n}}' = \hat{\mathbf{n}} - \nabla(\hat{\mathbf{n}}\cdot\beta)$$

Modulation and aberration of the anisotropies

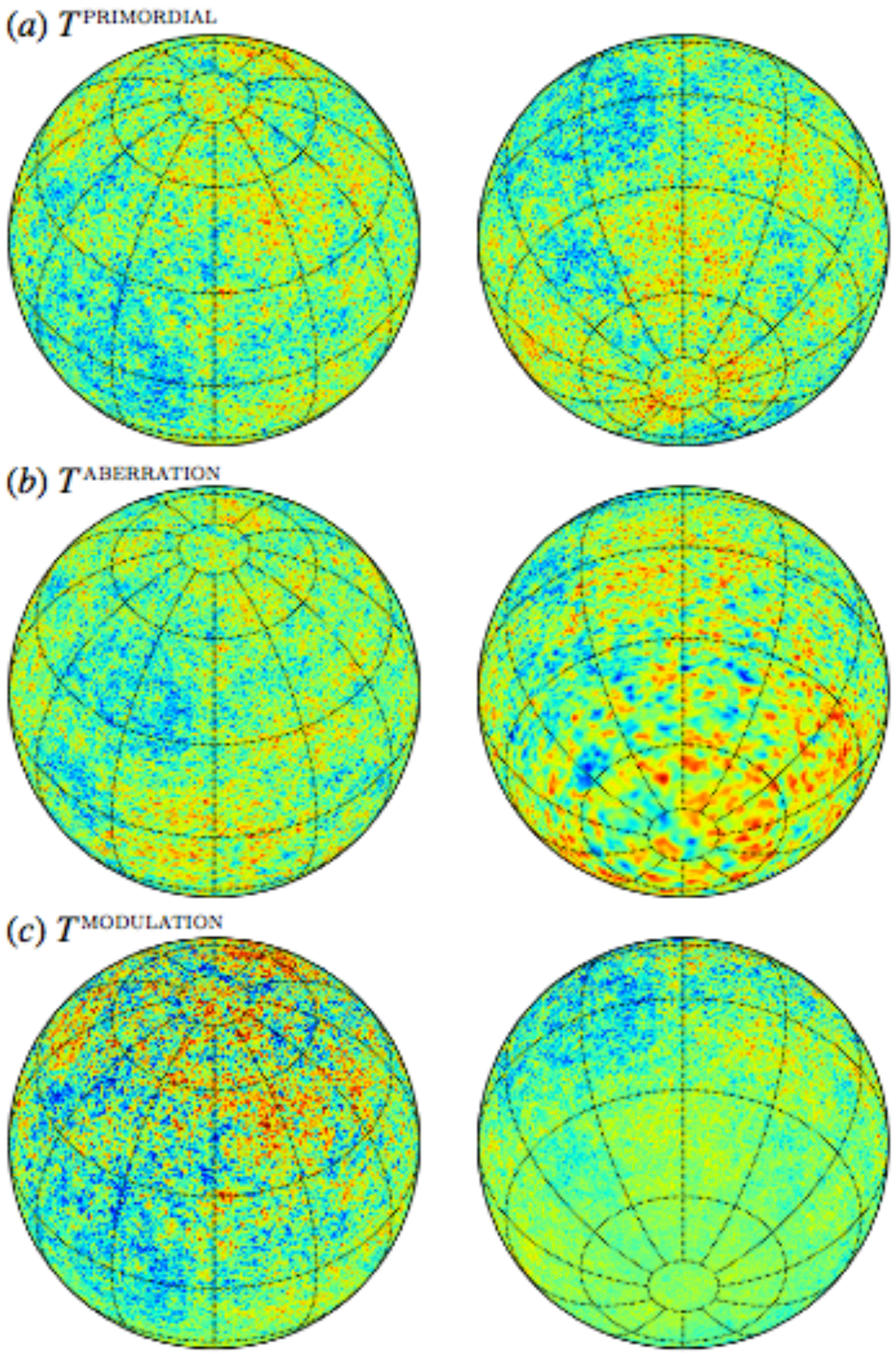
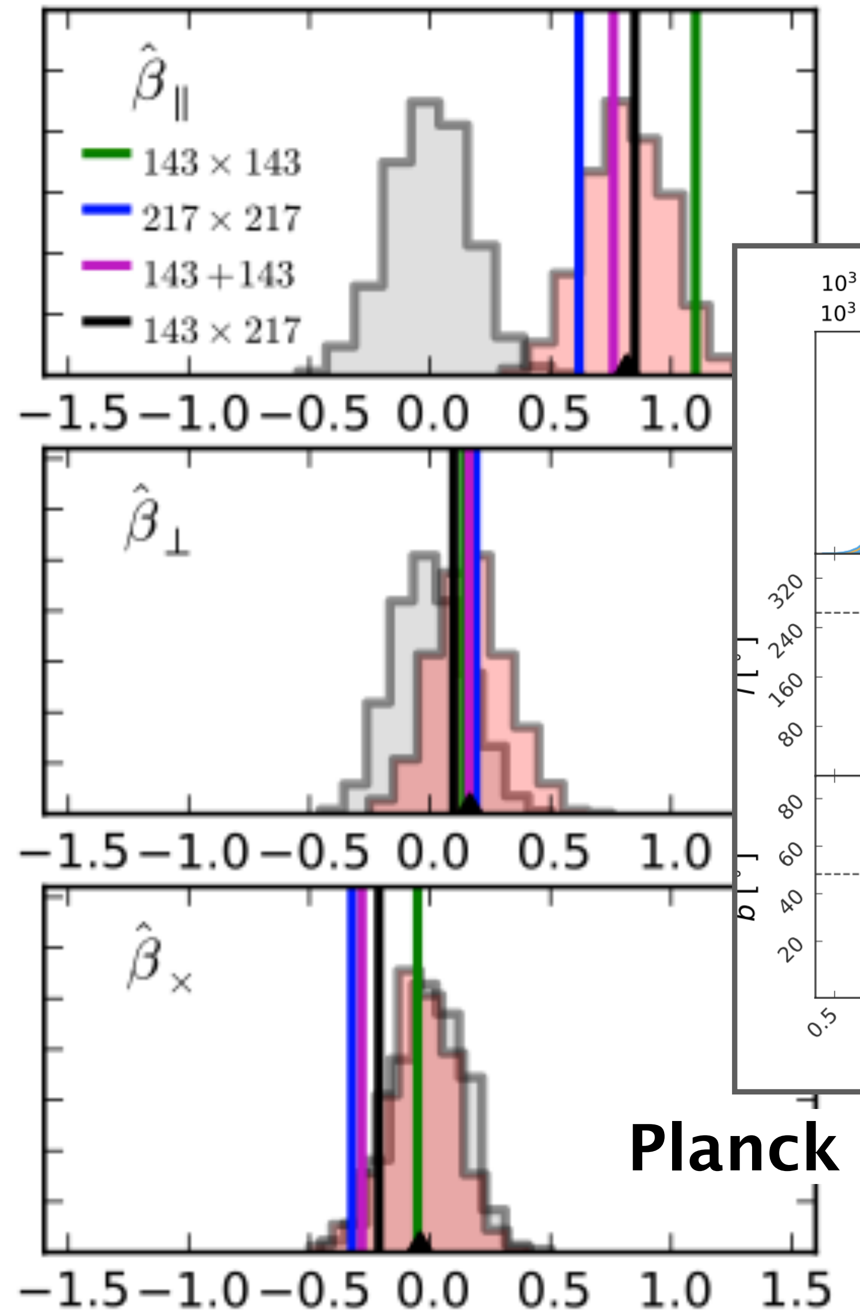


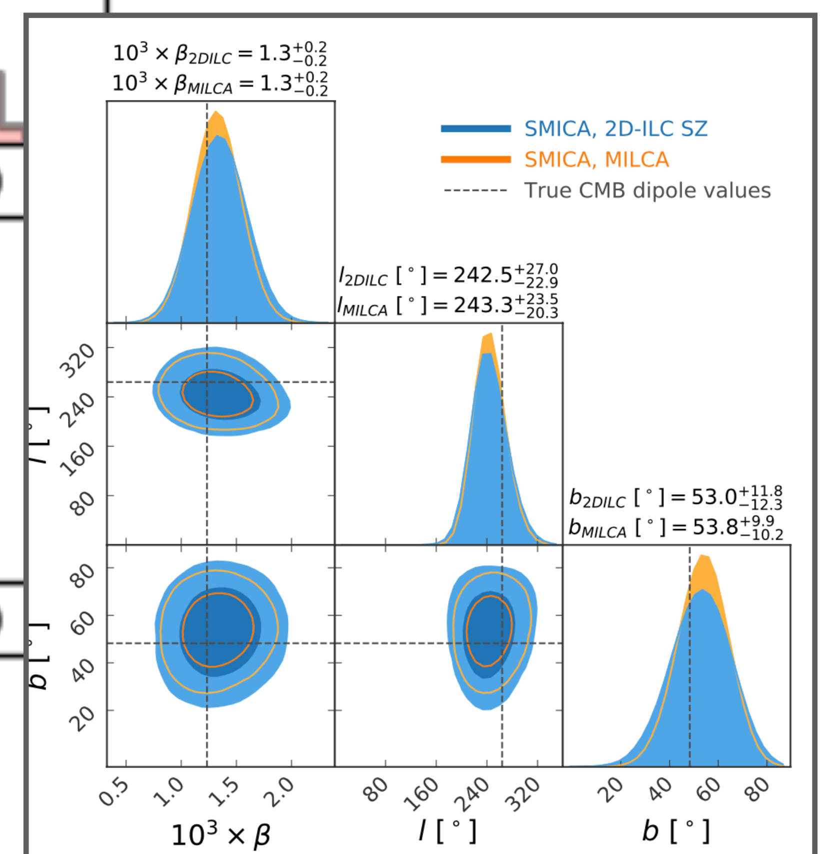
Fig. 1. Exaggerated illustration of the Doppler aberration and modulation effects, in orthographic projection, for a velocity $v = 260\,000\,\mathrm{km\,s^{-1}} = 0.85c$ (approximately 700 times larger than the expected magnitude) toward the northern pole (indicated by meridians in the upper half of each image on the left). The aberration component of the effect shifts the apparent position of fluctuations toward the velocity direction, while the modulation component enhances the fluctuations in the velocity direction and suppresses them in the anti-velocity direction.

Both these are level ~β effects. Modulation is the brightening/dimming of temperature anisotropies, and aberration is a relativistic shift in their angular size. These were measured by *Planck* in 2013 to about 4σ.



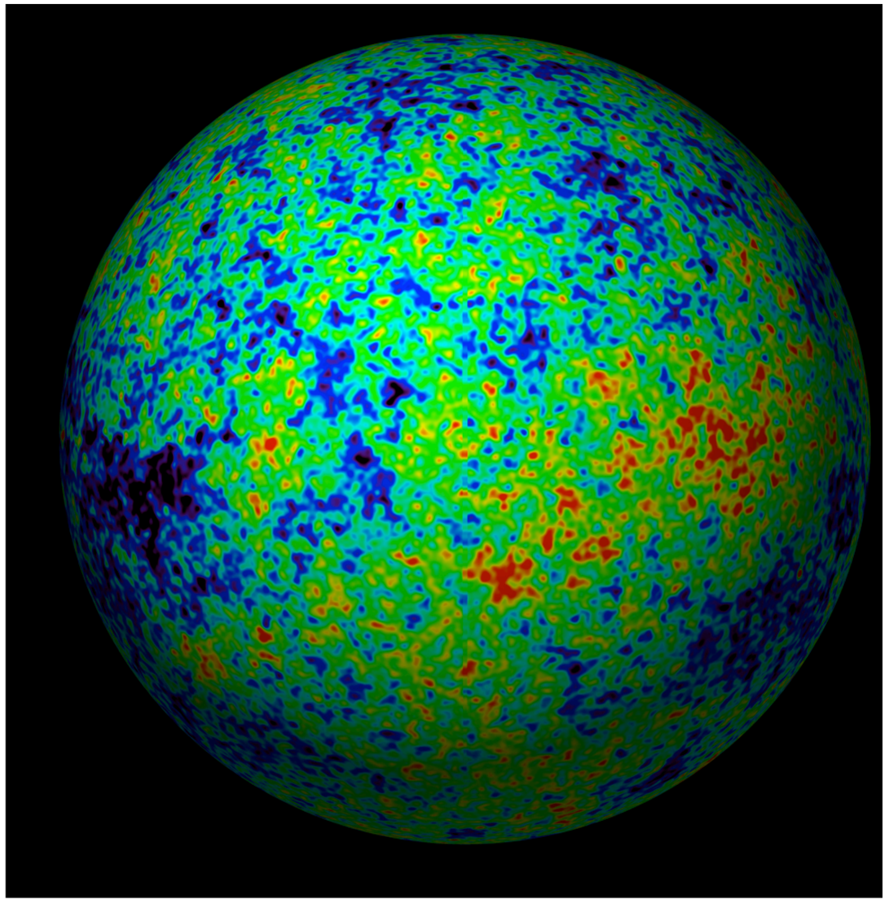
Planck collaboration (2013)

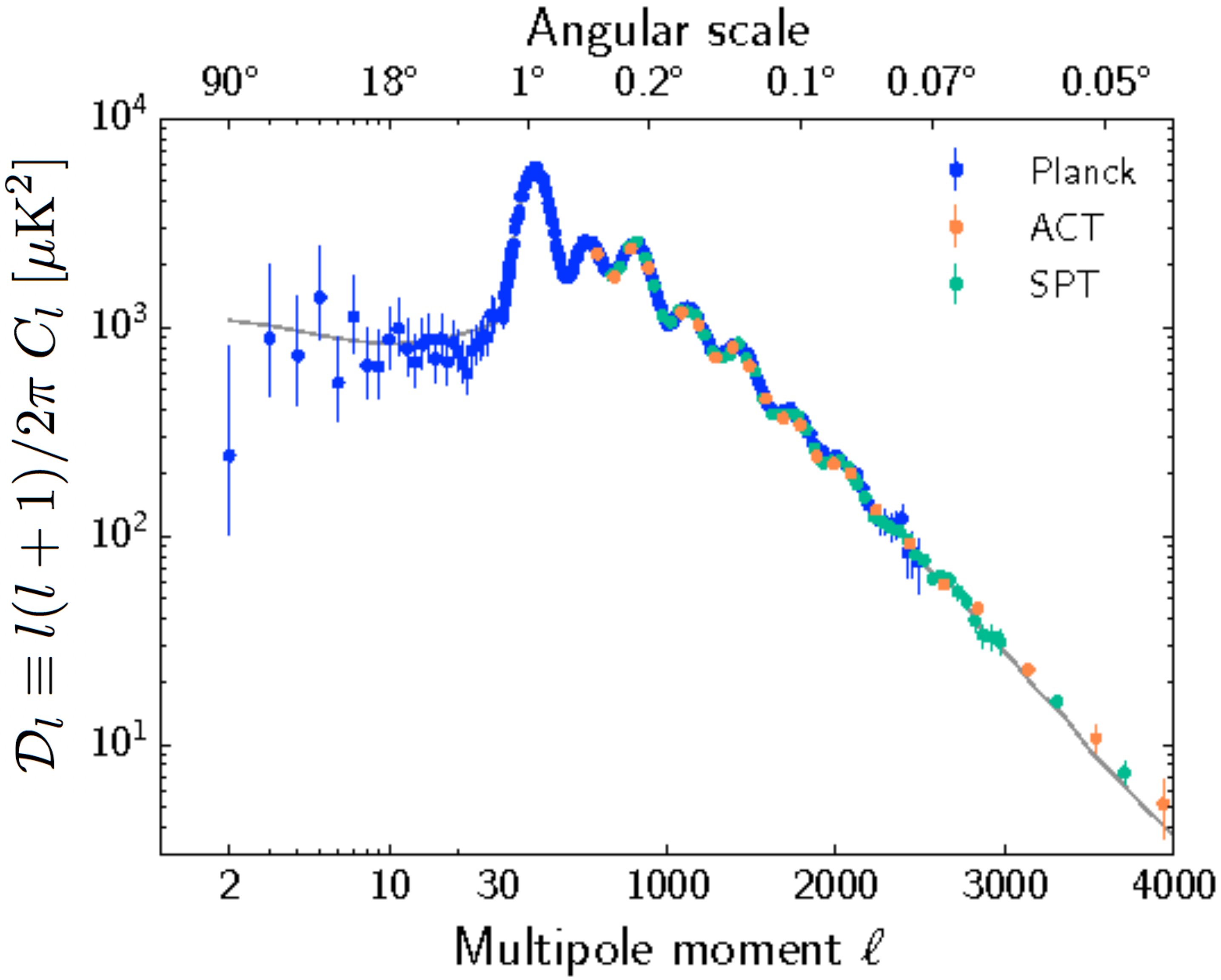
Now also measured through the SZ signal modulation in the sky



Planck collaboration (2020)

Temperature anisotropies: The CMB power spectrum





Recap: Gaussian random fields and power spectrum

A random field is the generalization of random variables at different points in space (and/or time). Denoting such fields by $\phi(\mathbf{x})$, their statical properties are described by the correlation function of the form $\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle$. In general there can also be three-point and higher orders correlation terms. But these vanish for Gaussian random fields!

For Gaussian random fields, the variables take on the Gaussian probability distribution. In this case, the **2-point correlation function** describes its statistical properties completely! We can also take advantage of the homogeneity and isotropy of the fields, to take Fourier transform of the field, and define the power spectrum, *P(k)*, which only depends on the amplitude of the wave vector ***k***. Thus $\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = P(k)$

$$P(k) = \int d^3r \; \xi(\vec{r}) \; e^{i\vec{k}\cdot\vec{r}}$$

$$\xi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} P(k) e^{-i\vec{k}\cdot\vec{r}}$$

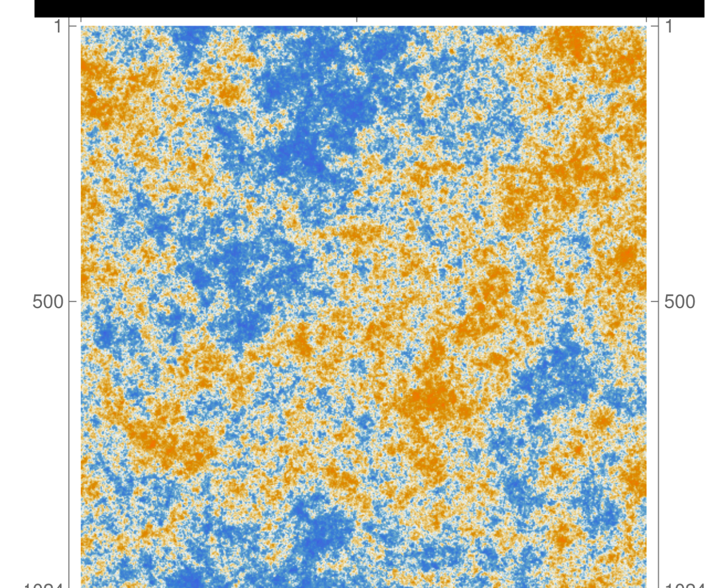
$$\sigma^2 = \int \frac{d\vec{k}}{(2\pi)^3} P(k)$$

Gaussian random fields are fully described by their 2nd order moment.

In real (map) space:
2-pt correlation function

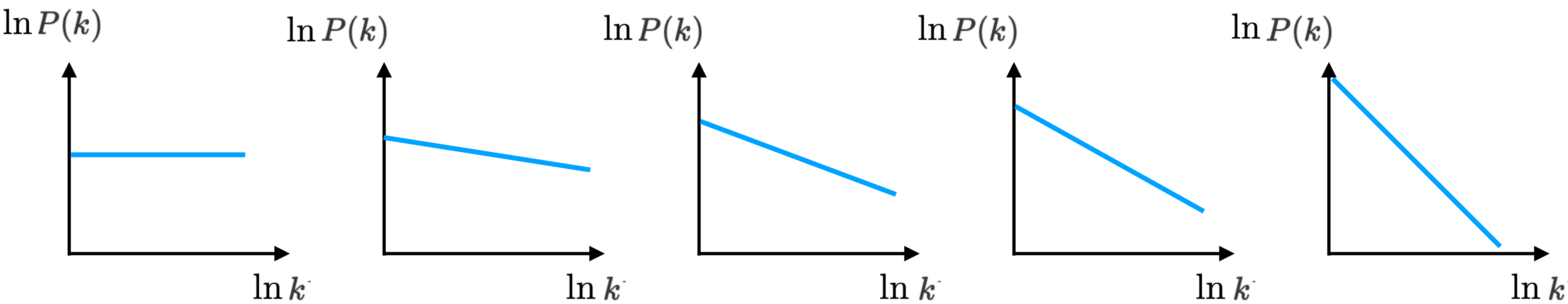
In fourier space:
Power spectrum

$$\xi(\vec{r}_1, \vec{r}_2) = \xi(|\vec{r}_1 - \vec{r}_2|) = \langle f(\vec{r}_1) f(\vec{r}_2) \rangle$$

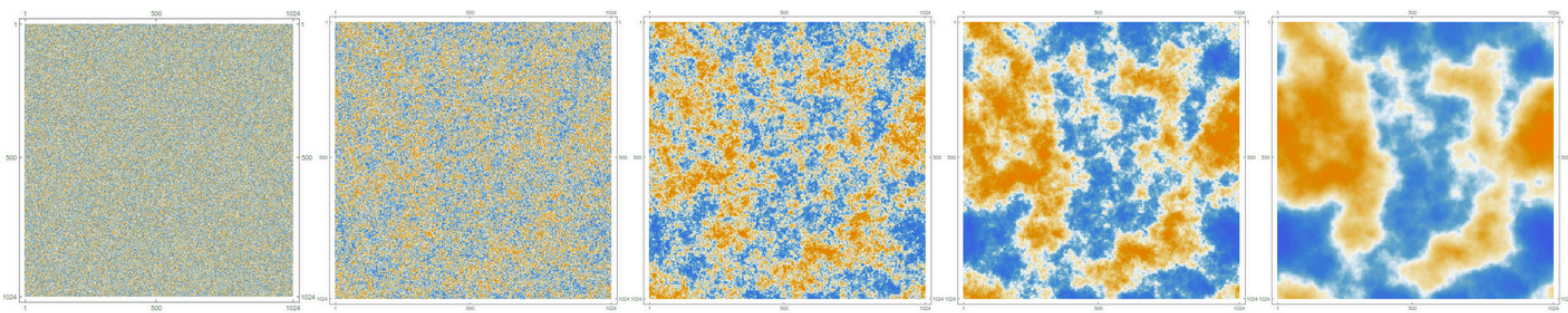


$$(2\pi)^3 P(k_1) \, \delta_D(\vec{k}_1 - \vec{k}_2) = \langle \hat{f}(\vec{k}_1) \hat{f}^*(\vec{k}_2) \rangle$$

Recap: Gaussian random fields and power spectrum



Realizations for $P(k) = k^{-n}$ with $n \in \{0, 1, 2, 3, 4\}$ from left to right.



$P(k) = k^0$ $P(k) = k^{-1}$ $P(k) = k^{-2}$ $P(k) = k^{-3}$ $P(k) = k^{-4}$

See https://garrettgoon.com/gaussian-fields/

CMB temperature anisotropies

• The basic observable is the CMB intensity as a function of frequency and direction on the sky. Since the CMB spectrum is an extremely good black body with a fairly constant temperature across the sky, we generally describe this observable in terms of a temperature fluctuation

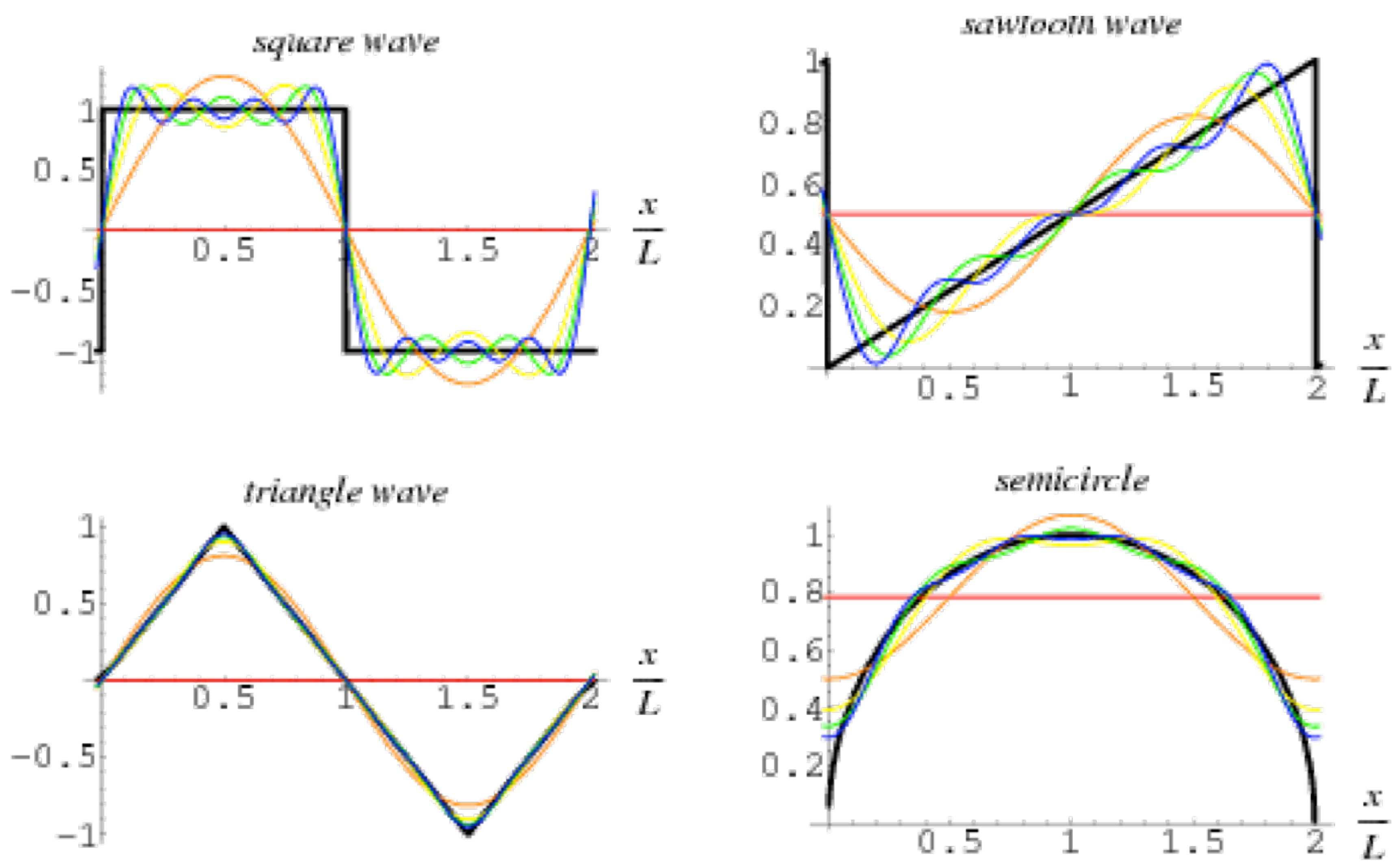
$$\frac{\Delta T}{T}(\theta, \phi) = \frac{T(\theta, \phi) - \bar{T}}{\bar{T}}$$

• The equivalent of the Fourier expansion on a sphere is achieved by expanding the temperature fluctuations in spherical harmonics

$$\frac{\Delta T}{T}(\theta, \phi) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_m^{\ell}(\theta, \phi)$$

$$\ell = 0, 1, \ldots \,, \quad m = -\ell, -\ell + 1, \ldots, \ell$$

Analogy: Fourier series



Sum sine waves of different frequencies to approximate any function.

Each term has a coefficient, or amplitude. Know these amplitudes and you know the function!

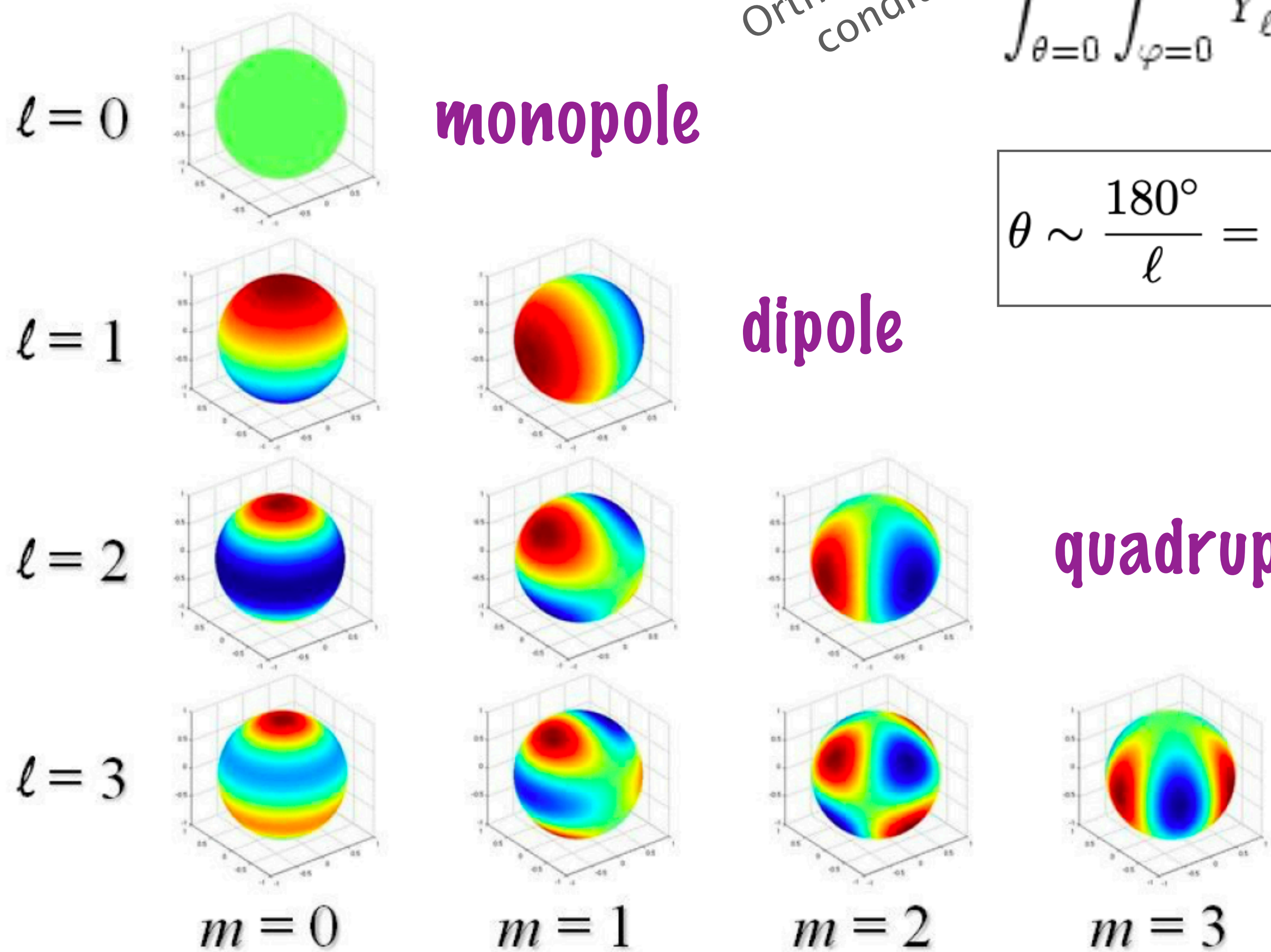
Spherical harmonics functions

Check out the wikipedia pages on Legendre polynomials and spherical harmonics

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\varphi}$$

Orthonormality condition

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} Y_\ell^m Y_{\ell'}^{m'*}\,d\Omega = \delta_{\ell\ell'}\delta_{mm'} \qquad d\Omega = \sin\theta\,d\varphi\,d\theta$$



monopole

dipole

quadrupole

octopole

$$\theta \sim \frac{180^\circ}{\ell} = \frac{\pi}{\ell}$$

$\vartheta \approx 360°/(2\ell+1) \approx 180°/\ell$

(imagine cutting up a sphere into $2\ell+1$ stripes)

A fully exact relation between multipole ℓ and 1D angle θ cannot be given. We use the above approximation.

Red implies positive values, blue is negative. **This plot shows only half the harmonic functions!**
$m = -l, -l+1, \dots, -1, 0, +1, \dots, l-1, l$ for each l

First two spherical harmonic functions

$$Y_0^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\,\sin\theta\,e^{-i\varphi}$$

$$Y_1^{0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\,\cos\theta$$

$$Y_1^{1}(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\,\sin\theta\,e^{i\varphi}$$

Visualizing the multipoles

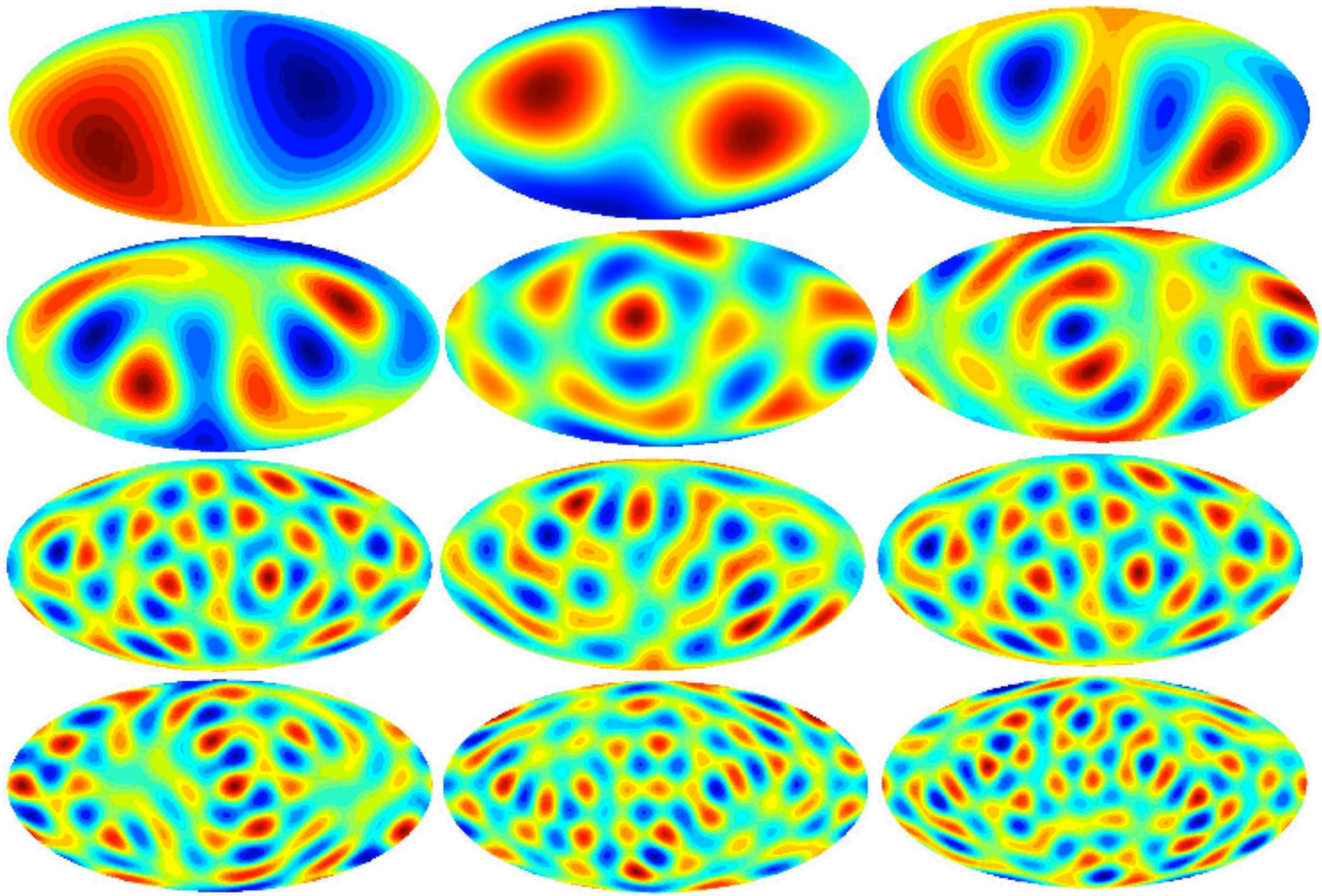
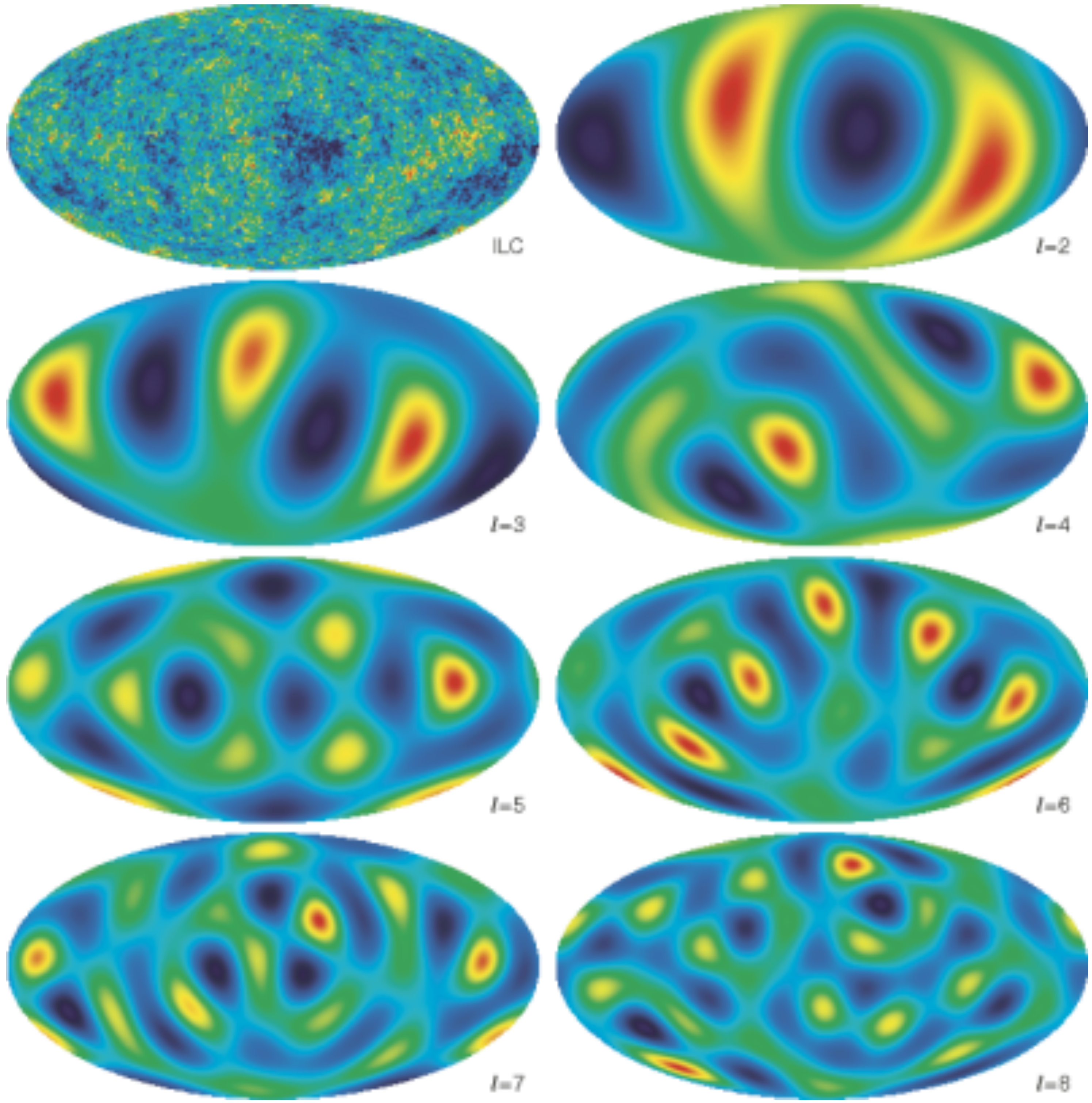


Figure 6: Randomly generated skies containing only a single multipole ℓ. Staring from top left: $\ell = 1$ (dipole only), 2 (quadrupole only), 3 (octupole only), 4, 5, 6, 7, 8, 9, 10, 11, 12. Figure by Ville Heikkilä.

Visualizing the multipoles (real CMB)

WMAP 2007 result



Spherical harmonics & power spectrum

CMB temperature anisotropies are expressed in terms of spherical harmonics

$$T(\hat{\mathbf{n}}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}^{T} Y_{\ell m}(\theta, \phi), \qquad \text{where} \quad T(\hat{\mathbf{n}}) = \delta T(\hat{\mathbf{n}})/T_0$$

We define the $\langle . \rangle$ operator as the ensemble average, over all possible realizations from the same underlying theory, and we can use that to define correlation between the a_{lm}-s

$$C_{\ell m \ell' m'}^{TT} \equiv \left\langle a_{\ell m}^{T} \, a_{\ell' m'}^{T*} \right\rangle \quad \Rightarrow \quad C_{\ell}^{TT} \delta_{\ell \ell'} \delta_{m m'} = \left\langle a_{\ell m}^{T} \, a_{\ell' m'}^{T*} \right\rangle$$

where we used the homogeneity and isotropy (i.e. rotational invariance) of the anisotropies. From this we can obtain the 2-point correlation function of the temperature anisotropies:

$$\begin{aligned}
\langle T(\hat{\mathbf{n}}) T(\hat{\mathbf{n}}') \rangle &= \left\langle \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}^{T} Y_{\ell m}(\theta, \phi) \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} a_{\ell' m'}^{T} Y_{\ell' m'}(\theta', \phi') \right\rangle \\
&= \left\langle \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |a_{\ell m}^{T}|^2 Y_{\ell m}(\theta, \phi) Y_{\ell m}(\theta', \phi') \right\rangle . \\
&= \frac{1}{4\pi} \sum_{\ell=1}^{\infty} (2\ell+1) C_{\ell}^{TT} P_{\ell}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}')
\end{aligned}$$

where we used the identity

$$P_{\ell}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}') = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{n}}) Y_{\ell m}(\hat{\mathbf{n}}')$$

Spherical harmonics & power spectrum

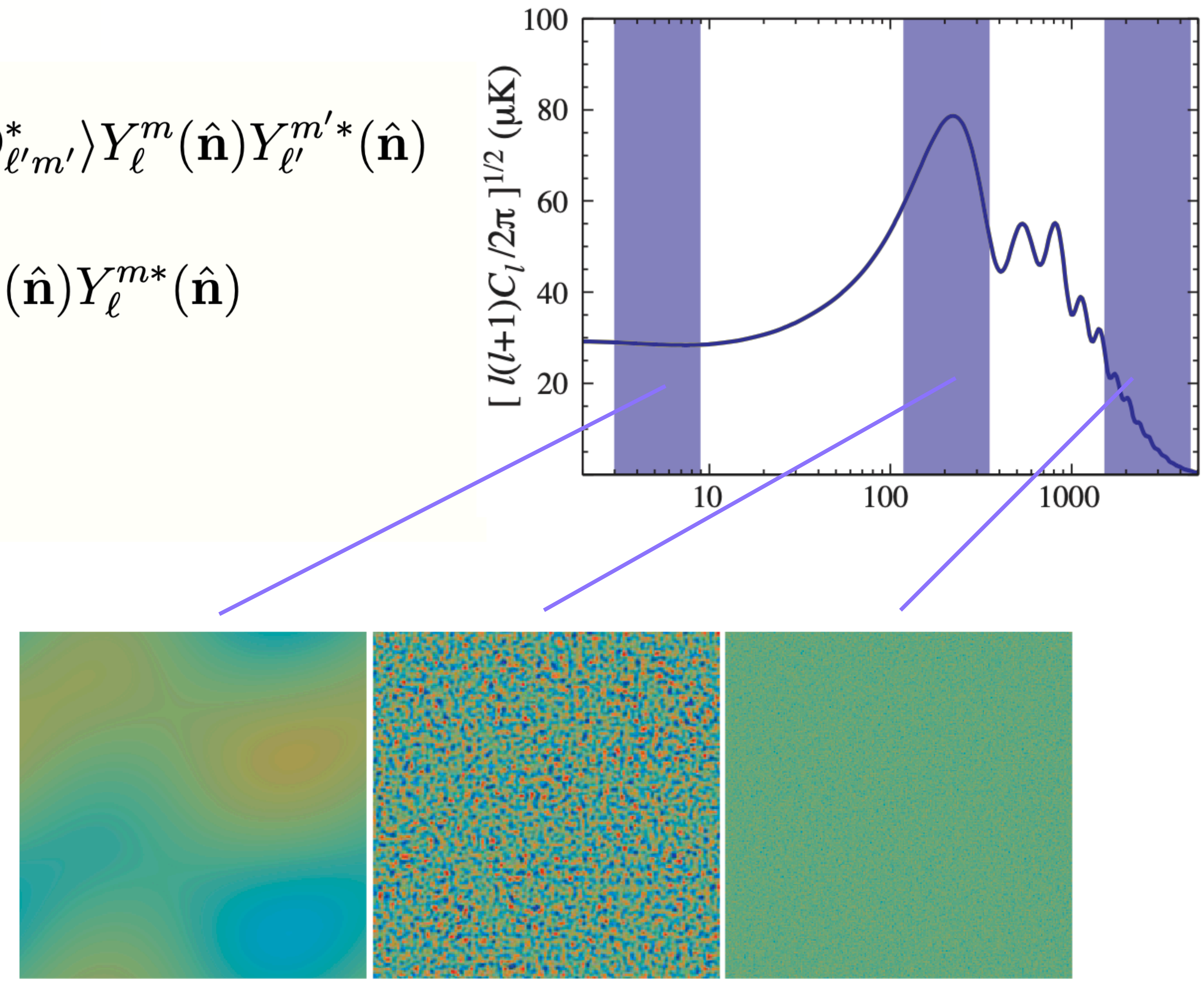
Therefore, from the definition of variance, we see that the angular power spectrum, C_l, gives the **variance of a temperature field** at a particular l-mode.

We define $\Theta(\hat{\mathbf{n}}) \equiv \dfrac{T(0,\hat{\mathbf{n}},t_0)-\bar{T}}{\bar{T}}$

$$
\begin{aligned}
\langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}})\rangle &= \sum_{\ell m}\sum_{\ell' m'} \langle \Theta_{\ell m}\Theta^*_{\ell' m'}\rangle Y_\ell^m(\hat{\mathbf{n}}) Y_{\ell'}^{m' *}(\hat{\mathbf{n}}) \\
&= \sum_\ell C_\ell \sum_m Y_\ell^m(\hat{\mathbf{n}}) Y_\ell^{m*}(\hat{\mathbf{n}}) \\
&= \sum_\ell \frac{2\ell+1}{4\pi} C_\ell
\end{aligned}
$$

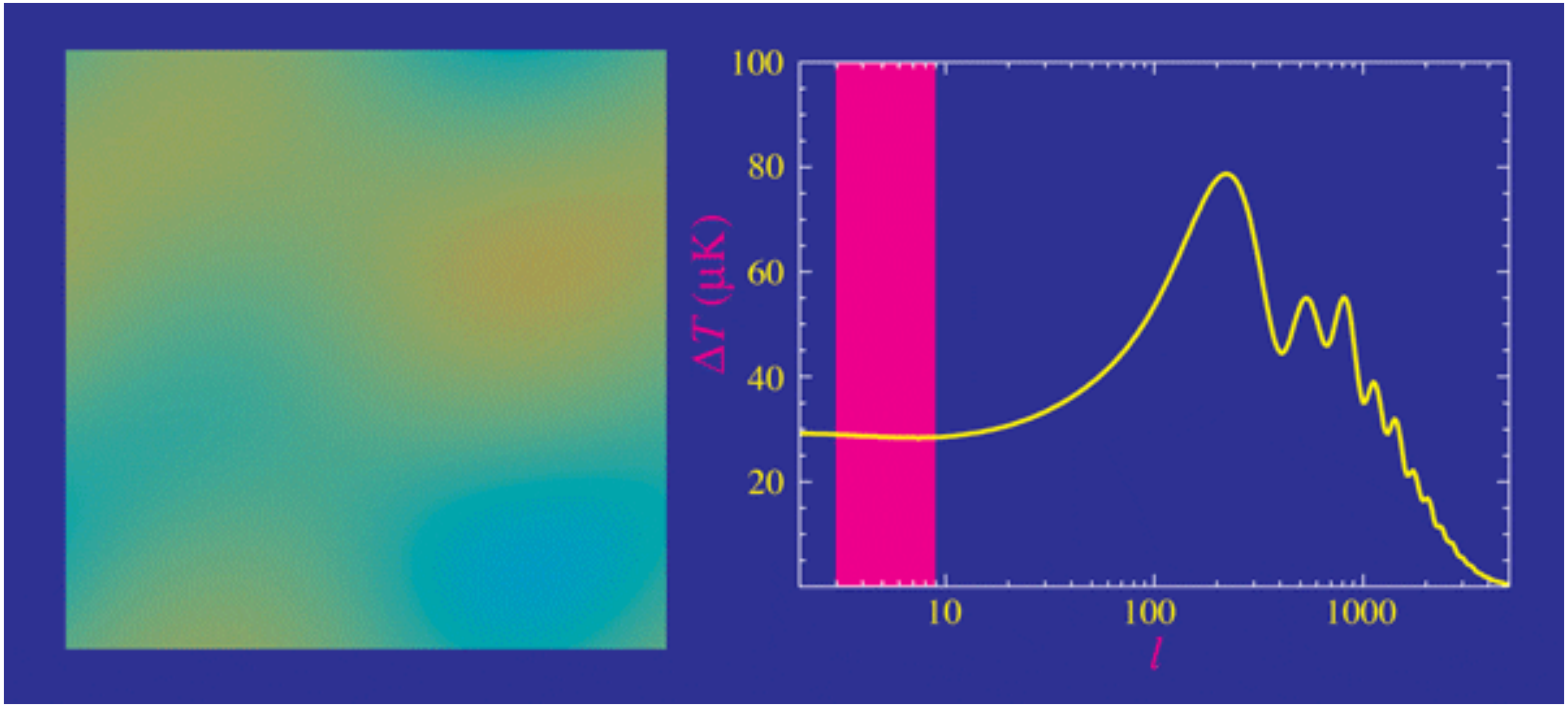
The variance of temperature anisotropies

$$\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell$$



Credit: Wayne Hu

Power at different scales



The variance of temperature anisotropies

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell$$

Credit: Wayne Hu
http://background.uchicago.edu/~whu/metaanim.html

Spherical harmonics summary

Temperature anisotropies are expressed in terms of spherical harmonics

$$\frac{\Delta T}{T}(\theta,\phi) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_m^\ell(\theta,\phi) \qquad a_{\ell m} = \int d\Omega Y_{\ell m}^*(\theta,\phi) \frac{\delta T}{T}(\theta,\phi)$$

Gaussian random field: Mean of the harmonic coefficients is zero, uncorrelated on different scales and angles

$$\langle a_{\ell m} \rangle = 0 \,. \quad \langle a_{\ell m} a^*_{\ell' m'} \rangle = 0 \qquad \text{if } \ell \neq \ell' \text{ or } m \neq m'$$

The expectation values of the coefficients give the power spectrum (sum over all m-modes for a given l)

$$C_\ell = \langle |a_\ell^m|^2 \rangle = \frac{1}{2\ell+1} \sum_m |a_\ell^m|^2$$

For a **Gaussian random field**, the C_l contains all the statistical information. Thus the cosmological analysis of CMB temperature anisotropies (and likewise for its polarization anisotropies, see later) consists of computing the C_l-s from the real sky and comparing those with the theoretical models (e.g., using Bayesian methods).

The correspondence between the 2-pt correlation function in real space and the angular power spectrum C_l

$$\xi(x) \equiv \langle f(n) f(n') \rangle = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos(\theta))$$

Spherical harmonics summary

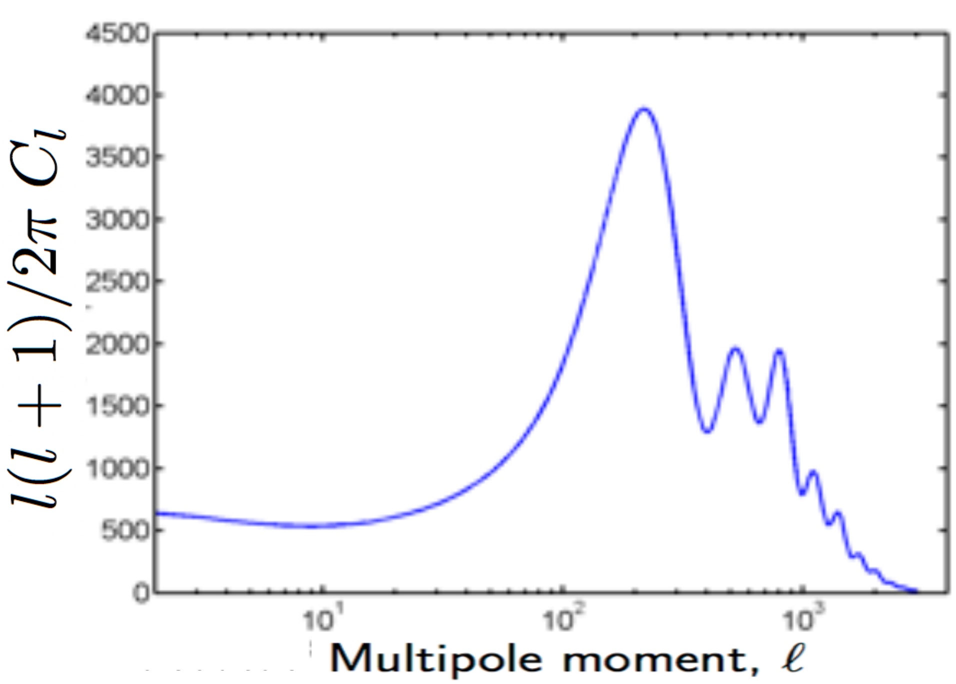
In real space *In harmonic space*

$$\xi_{\Theta\Theta}(\theta) = \langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}}')\rangle = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)C_\ell P_\ell \cos\theta,\ \ \hat{\mathbf{n}}.\hat{\mathbf{n}}' = \cos\theta \qquad C_\ell = \langle |a_\ell^m|^2\rangle = \frac{1}{2\ell+1}\sum_m |a_\ell^m|^2$$

l~180°/θ relates to the angular size of the pattern, and m relates to the direction on the sky. $\langle |a_{lm}|^2\rangle$ is independent of m (since CMB is isotropic).

The mean $\langle a_{lm}\rangle$ is zero (Gaussian random field), but the variance is non-zero! The variance is the sum of angular power on all scales.

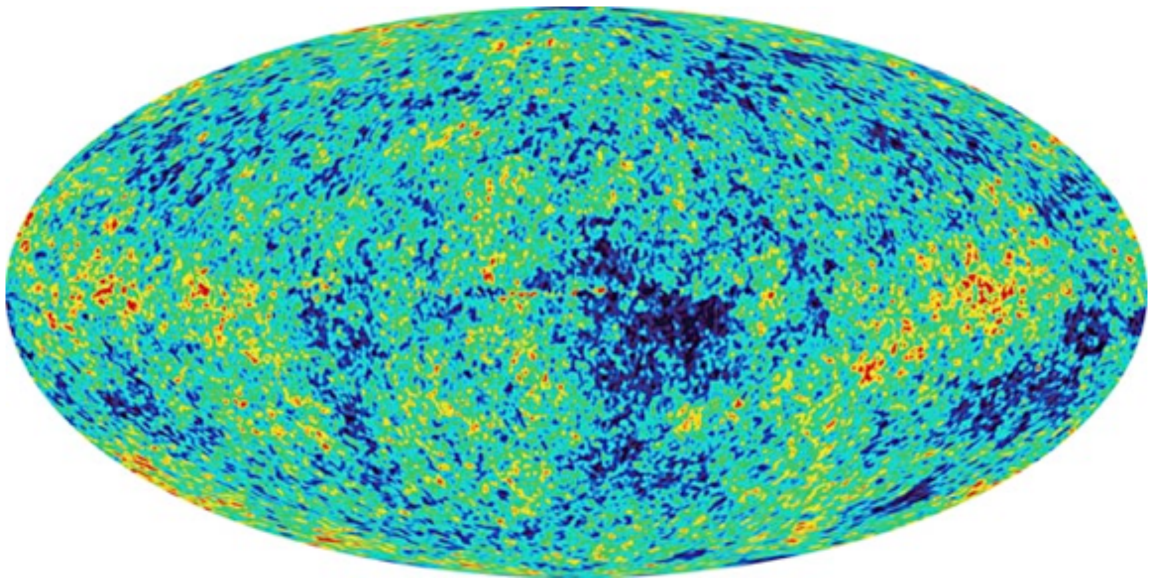
$$\left\langle \left(\frac{\delta T(\theta,\phi)}{T}\right)^2 \right\rangle = \left\langle \sum_{\ell m} a_{\ell m}Y_{\ell m}(\theta,\phi)\sum_{\ell' m'} a^*_{\ell' m'}Y^*_{\ell' m'}(\theta,\phi)\right\rangle$$
$$= \sum_{\ell\ell'}\sum_{mm'} Y_{\ell m}(\theta,\phi)Y^*_{\ell' m'}(\theta,\phi)\langle a_{\ell m}a^*_{\ell' m'}\rangle$$
$$= \sum_\ell C_\ell \sum_m |Y_{\ell m}(\theta,\phi)|^2 = \sum_\ell \frac{2\ell+1}{4\pi}C_\ell\,,$$

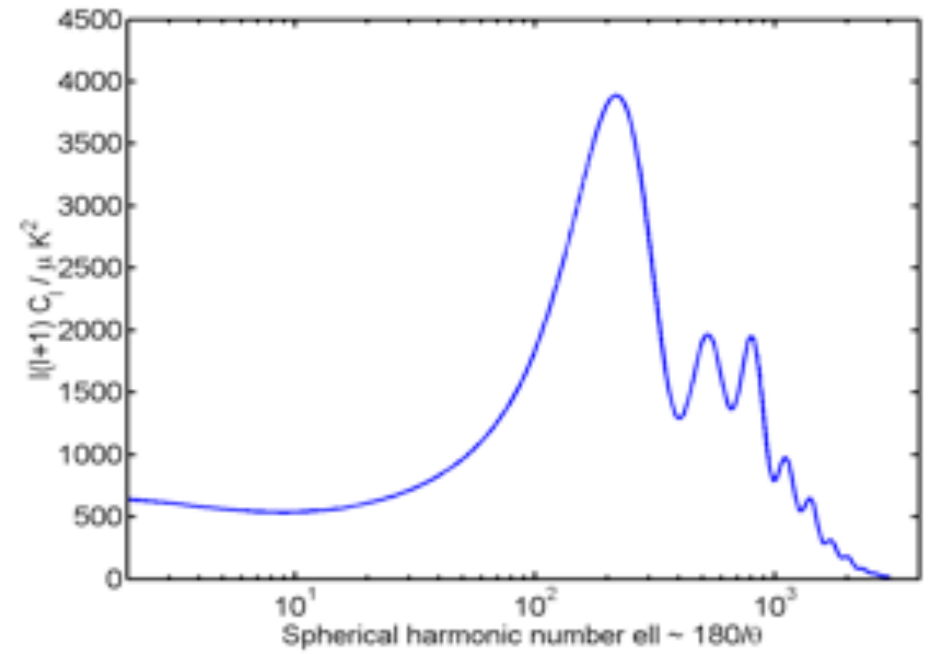


It is customary to plot this variance as $l(l+1)C_l/2\pi$ vs l

This quantity is expected to be a constant if temperature anisotropies were generated from primordial density perturbations alone (Sachs-Wolfe effect — see later).

CMB sky → Power spectrum





Computing the power spectrum from sky temperature fluctuations should then be easy! You just need to Fourier transform the map and calculate the coefficients a_{lm} and then take the statistical average. *In practice*, of course, this is numerically challenging for maps with millions of pixels on the sky.

$$\frac{\Delta T}{T}(\theta,\phi) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_m^{\ell}(\theta,\phi)$$

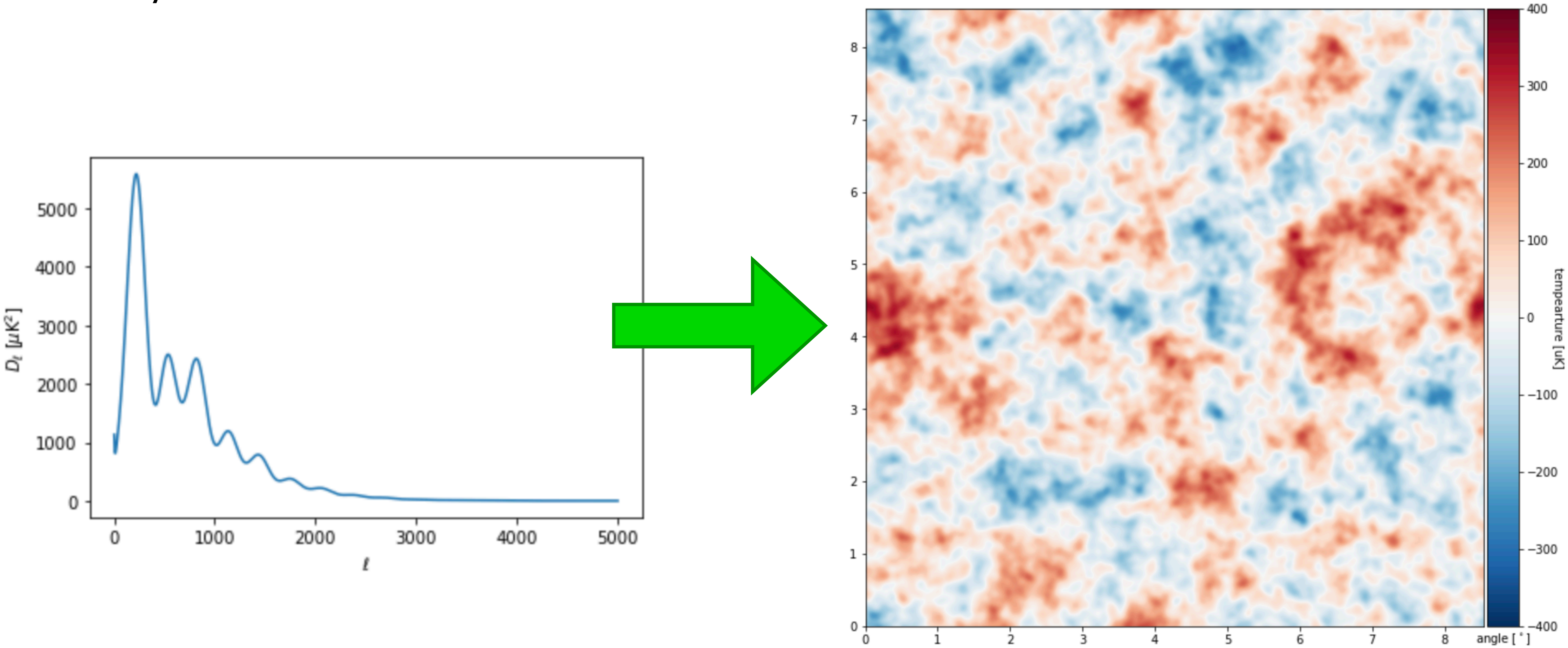
$$c_\ell = \langle |a_{\ell m}| \rangle^2$$

What we're plotting is the amplitude of fluctuations on each scale. It is also called the TT power spectrum, to denote the autocorrelation and to distinguish it from the polarization spectra (later in the class).

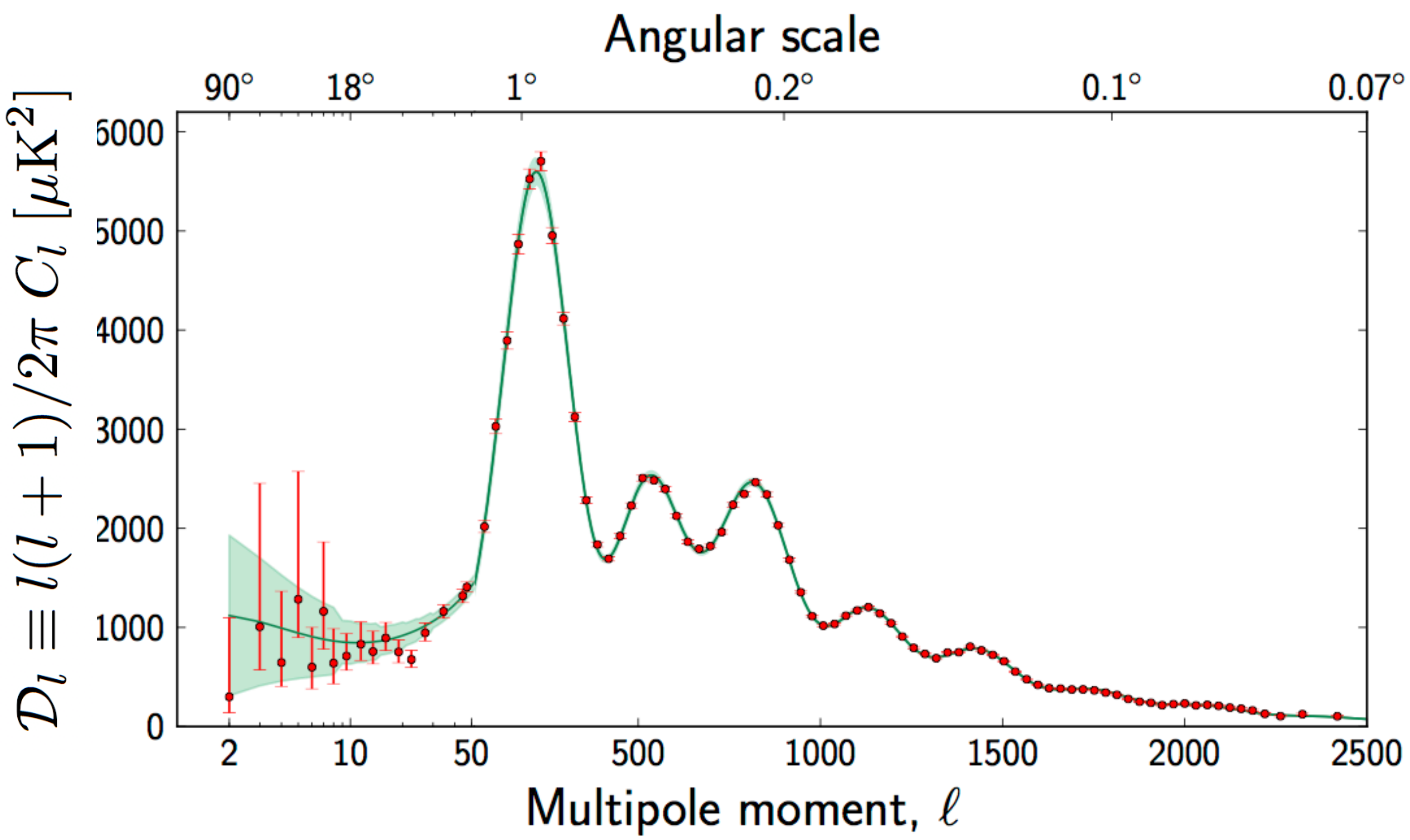
Exercise!

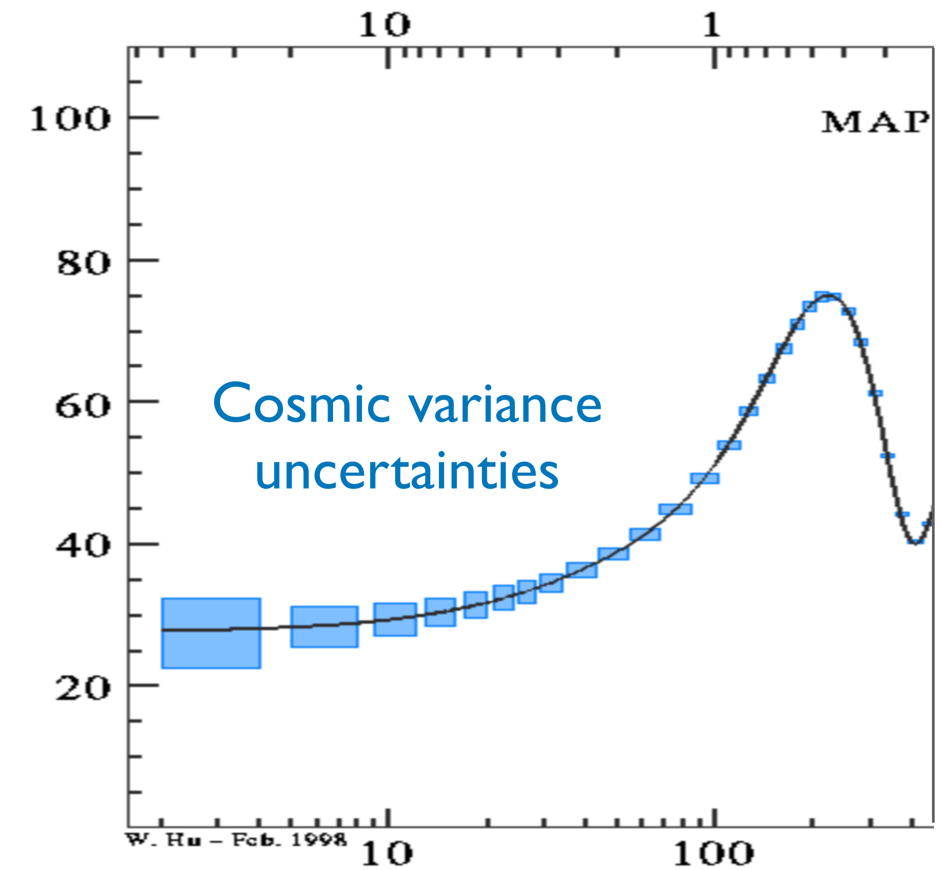
Power spectrum → CMB sky

We give you this as an exercise (using iPython notebook) to create a CMB sky patch from its power spectrum. Use an online CMB power spectrum generator (as with CAMB in the NASA LAMBDA site: https://lambda.gsfc.nasa.gov/toolbox/camb_online.html) to get your C_l vs l, make a 2D version of it, and Fourier transform it to create your own CMB sky!



Cosmic variance





We only have one Universe, so we are intrinsically limited to the number of independent m-modes we can measure – there are only (2l + 1) of these for each multipole.

We obtained the following expression for the power spectrum:

$$C_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \langle |a_{lm}|^2 \rangle$$

We can see (2l+1)C_l follows a χ^2 distribution. It has (2l+1) degrees of freedom, thus its variance is simply 2(2l+1). Therefore we get , **for any given C_l**

$$\Delta C_\ell = \sqrt{2/(2\ell+1)}$$

How well we can estimate an average value from a sample depends on how many points we have on the sample. This cosmic variance is an **unavoidable** source of uncertainty when constraining models!

Cosmic variance formal derivation

- We only have access to our sky, not the ensemble average
- There are $2\ell + 1$ m-modes of given ℓ mode, so average

$$\hat{C}_\ell = \frac{1}{2\ell+1}\sum_m \Theta^*_{\ell m}\Theta_{\ell m}$$

- $\langle \hat{C}_\ell \rangle = C_\ell$ but now there is a cosmic variance

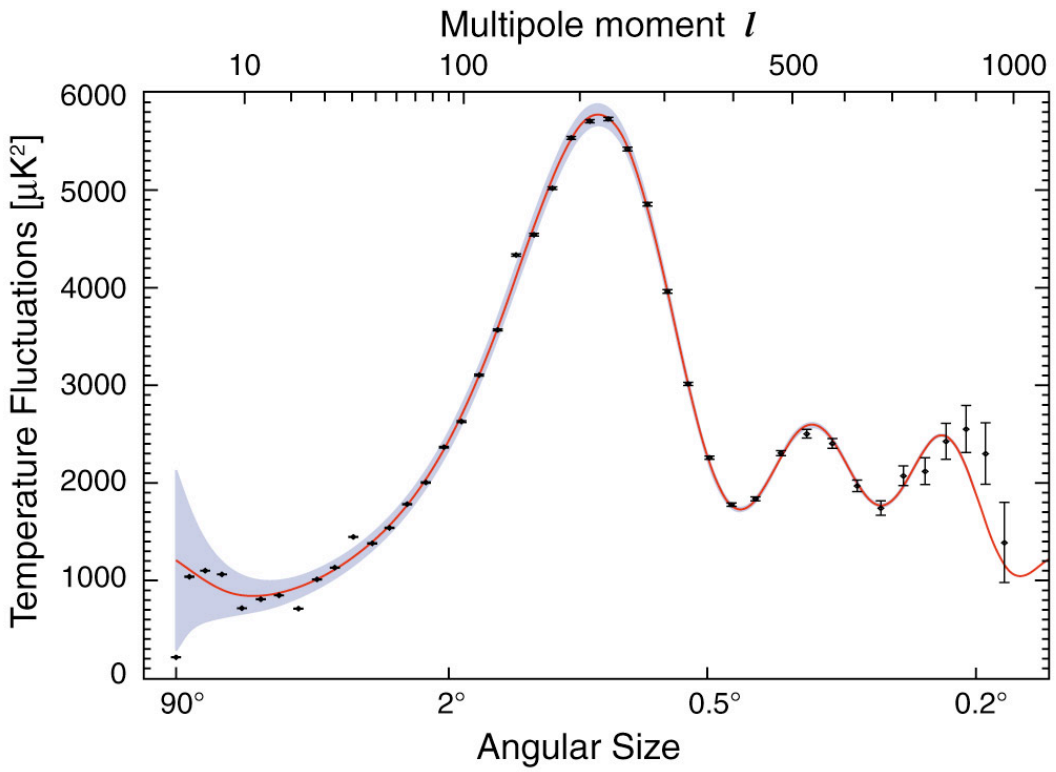
$$\sigma^2_{C_\ell} = \frac{\langle (\hat{C}_\ell - C_\ell)(\hat{C}_\ell - C_\ell)\rangle}{C_\ell^2} = \frac{\langle \hat{C}_\ell \hat{C}_\ell \rangle - C_\ell^2}{C_\ell^2}$$

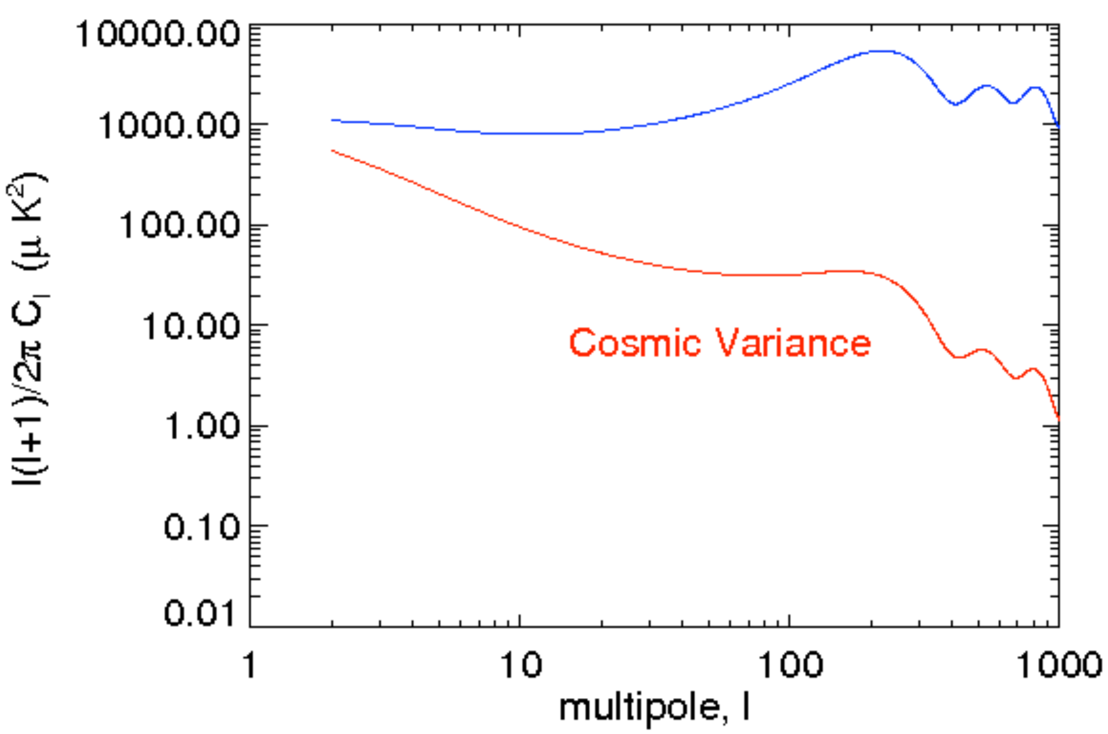
$$= \frac{1}{(2\ell+1)^2 C_\ell^2}\langle \sum_{mm'} \Theta^*_{\ell m}\Theta_{\ell m}\Theta^*_{\ell m'}\Theta_{\ell m'}\rangle - 1$$

$$= \frac{1}{(2\ell+1)^2}\sum_{mm'}(\delta_{mm'} + \delta_{m-m'}) = \frac{2}{2\ell+1}$$

- Note that the distribution of $\hat{C}_\ell$ is that of a sum of squares of Gaussian variates
- Distributed as a χ^2 of $2\ell + 1$ degrees of freedom
- Approaches a Gaussian for $2\ell + 1 \to \infty$ (central limit theorem)

Cosmic variance & sample variance





- **Cosmic variance**: on scale *l*, there are only *2l+1* independent modes

$$\Delta C_l = \sqrt{\frac{2}{2l+1}} C_l$$

- If the fraction of sky covered is *f*, then the errors are increased by a factor $1/\sqrt{f_{sky}}$ and the resulting variance is called **sample variance** (for example, $f_{sky} \sim 0.65$ for the *Planck* satellite)

$$(\Delta C_l)^2 = \frac{2}{(2\ell+1) f_{\rm sky}(\ell)} (C_\ell + N_\ell)^2$$

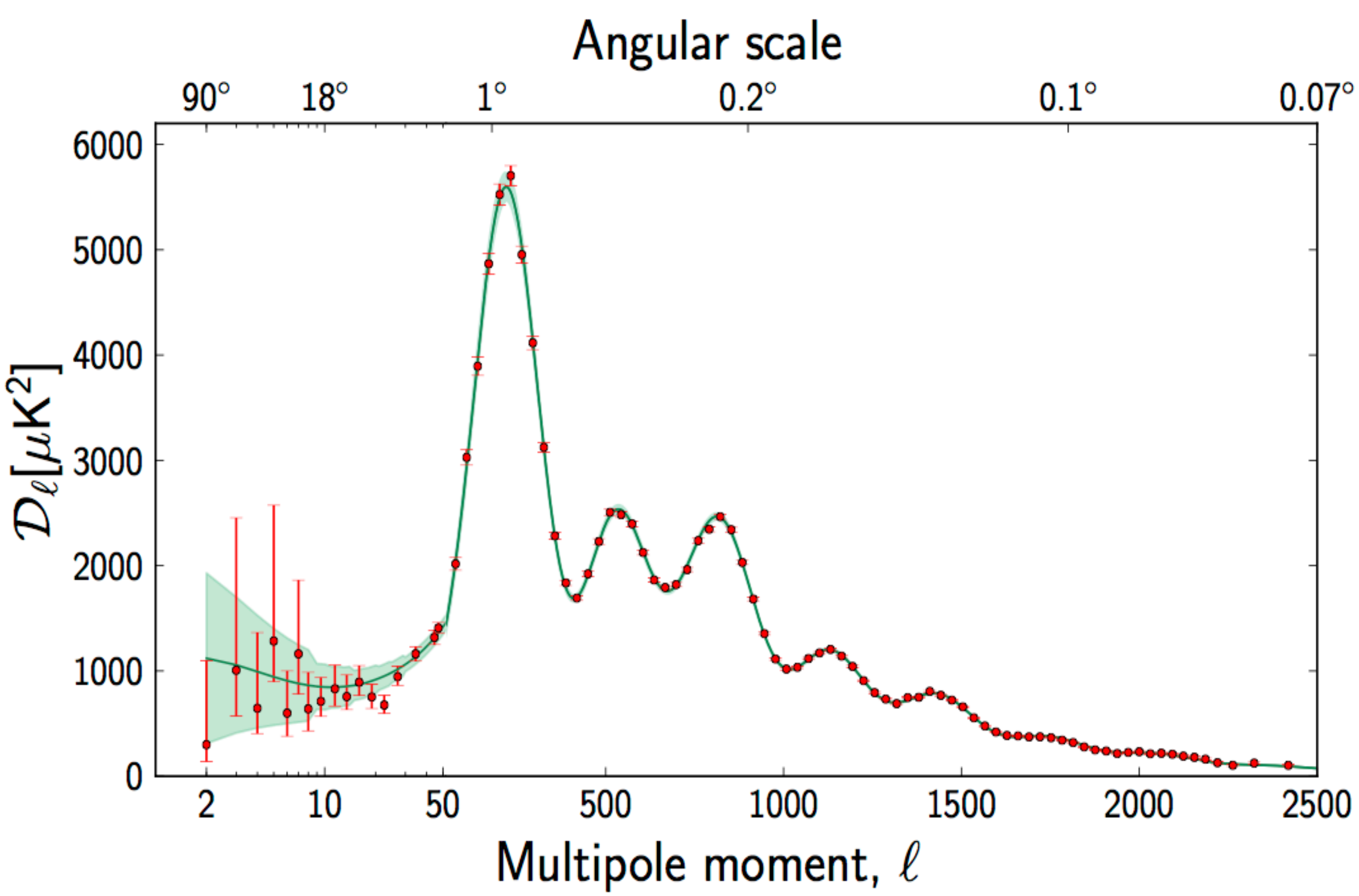
total variance

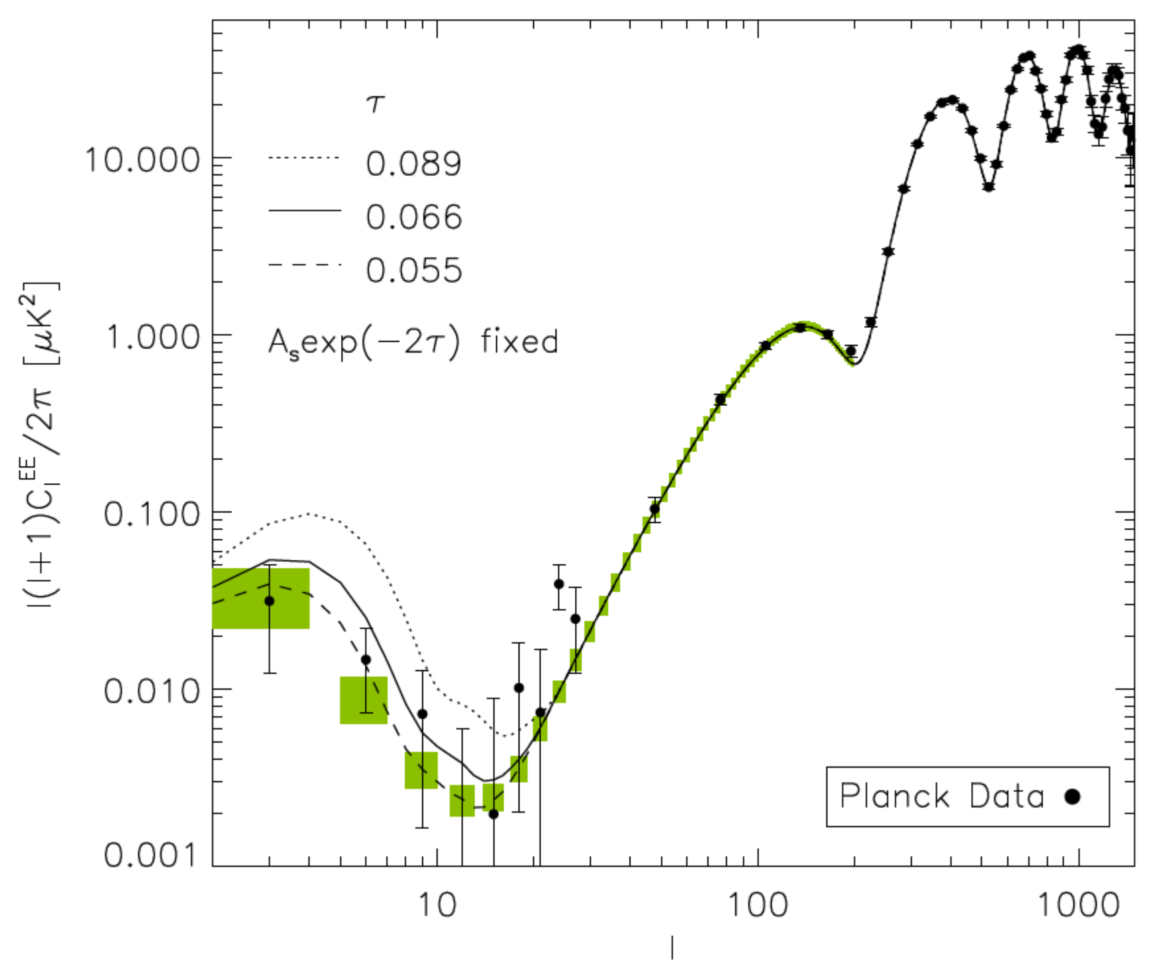
detector noise

Cosmic variance limits on the current CMB data

Current Planck temperature power spectrum (TT) is already cosmic variance limited, out to the angular scales where primordial effects are dominant. Hence, there is no gain to be made by measuring the CMB temperature anisotropies with more precision.

Within the next 4-5 years, LiteBIRD satellite will make cosmic variance-limited measurement of the EE mode of polarization power (highly useful for measuring the universe's optical depth caused by reionization, for example). And BB mode of polarization is fully unconstrained at the large scale. So there is plenty to discover!





Questions?

Feel free to email me or ask questions in our eCampus Forum