Homework 2 (hand in by 28th Oct 2019)

1. **Quickies** (2 points)
   (a) Write down the expression for comoving distance \( \chi \) for a given cosmological model and redshift \( z \).
   (b) Write down the expression for the luminosity distance in terms of the comoving angular diameter distance \( f_K \). How does \( f_K(\chi) \) look like for different cosmological models?

2. **Luminosity Distance** (6 points)
   (a) In a generic unperturbed FRW Universe, the luminosity distance can be Taylor expanded as
   \[
   D_L = \frac{c}{H_0} \left[ a_0 + a_1 z + a_2 z^2 + O(z^3) \right].
   \] (1)
   Find the values for the coefficients \( a_0, a_1, a_2 \).
   (b) Obtain the data points for the luminosity distance vs. redshift plot from the webpage: [http://cosmocalc.icrar.org](http://cosmocalc.icrar.org) and compare to Eq. (1). At what redshift does this second-order expansion break down?

3. **Volume elements in a FLRW-Universe** (3 points)
   Consider a volume element at redshift \( z \), consisting of a cylinder observed to subtend a solid angle \( d\omega \) at the observer and which is bounded in redshift at \( z - dz/2 \) and \( z + dz/2 \). Calculate its proper volume \( dV \).

4. **The future of the Universe** (4 points)
   The Friedmann equation can be written in the form
   \[
   H(a) = H_0 \sqrt{\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_{DE} a^\alpha},
   \] (2)
   where the dark energy density is allowed to change with \( a \).
   (a) For which values of \( \alpha \) does the dark energy term eventually become dominant?
   (b) Consider an epoch of the Universe when the matter, radiation and curvature density can be neglected. For this scenario, solve the Friedmann equation with the initial condition \( a(t = 0) = a_0 \). How does the scale factor \( a \) evolve with time for different values of \( \alpha \)? What happens if \( \alpha > 0 \)?
5. **Orbital period** (2 points)

Consider a spherical object of mass $M$ and radius $R$.

(a) Obtain the expression for the circular orbital period around this object.

(b) At what mean density is the orbital period equal to the Hubble time?