Gravity and thermodynamics: a new point of view in the analysis of dynamical evolution of globular clusters

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Preliminary considerations

From observations of glubular cluster (GC) density profiles we can deduce:

- an empirical density law for GCs (King, 1962);
- a unique sequence of models for GC evolution (King, 1966);
- that thermodynamics plays a central role in the gravitational equilibrium and stability of the clusters, being binary relaxation time shorter than the age of such systems $(\tau_{relax} < t_{age})$;
- that the evolution of GCs can be described as a sequence of thermodynamic equilibria where only parameters characterizing the distribution function change, like in a continuous thermodynamic transformation (Horwitz & Katz, 1977).

From the necessity to develop a theoretical framework taking into account thermodynamic equilibrium during the evolution of GCs we need to develope:

- a new concept of thermodynamic equilibrium based on Fokker-Planck approximation for collisions among stars, where the velocity distribution function does not change during the evolution.

Thermodynamic equilibrium is due to two competitive phenomena:

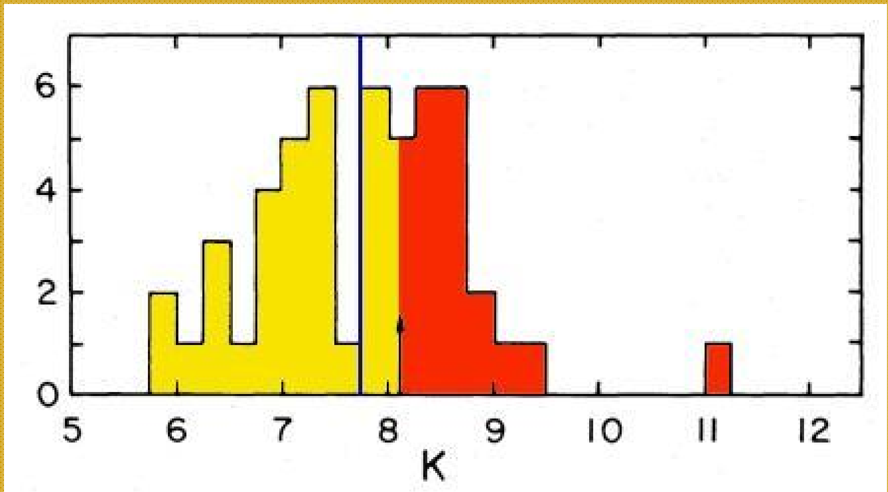
- collisions which drive the distribution towards a Boltzmann form;
- evaporation by tidal interactions induced by hosting galaxy, removing continuously the stars with large kinetic energy from the cluster.

→ ***The distribution maintains its form and may be considered in thermodynamic equilibrium: WE CAN APPLY MECHANICAL STATISTICS ON GCs***

A forty-years open question

Data analysis (Katz)

Gaussian assumption (Katz, 1980)



K values of 51 GCs from Peterson data (1976)

$$K = \frac{1}{2}\left(\frac{v_{e,0}}{v_{d,0}}\right)^2$$

$$K_{cr} = 8.1 \quad (W_0 = 7.4)$$

$$K_{rms} = 7.81 \quad (W_0 = 6.9)$$

K_{cr} and K_{rms} do not coincide. They should?

What happens if considering W_0?

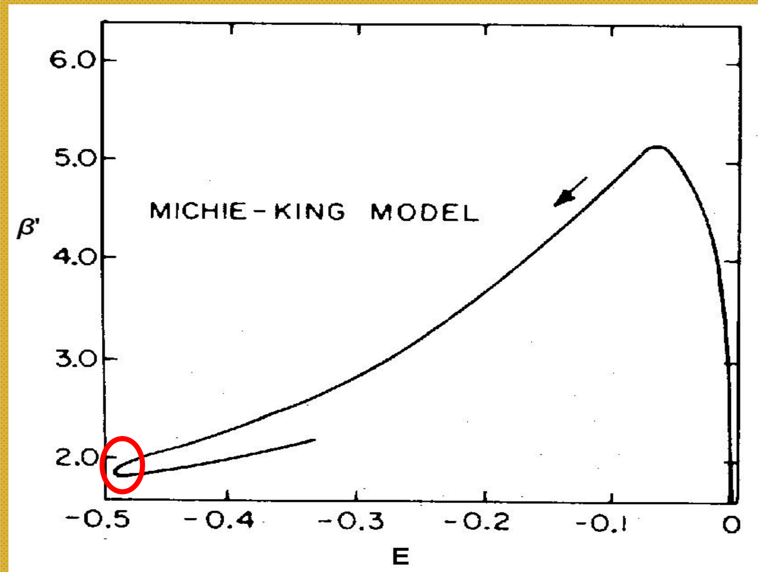
King Model

Stability range for classical single-mass King models

$$0 \le W_0 \le 7.4$$

Virial Theorem

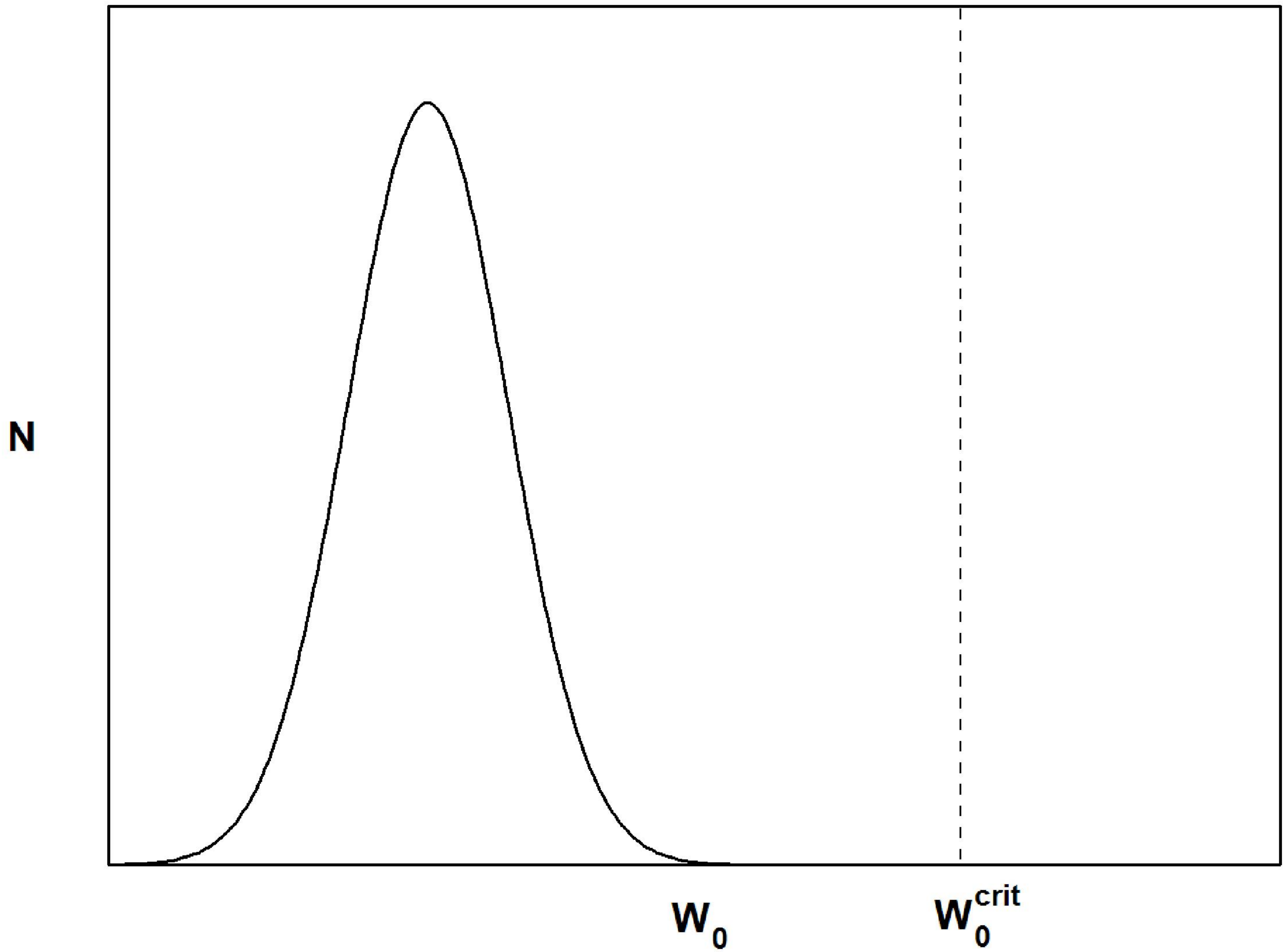
$$2E_K + E_{gr} = 0$$



Single mass equilibrium configurations

critical point:	inverse temperature:	energy:
W_0=7.4	β′ =m/kT	E=-K

Considering the evolution of GCs W_0-distribution



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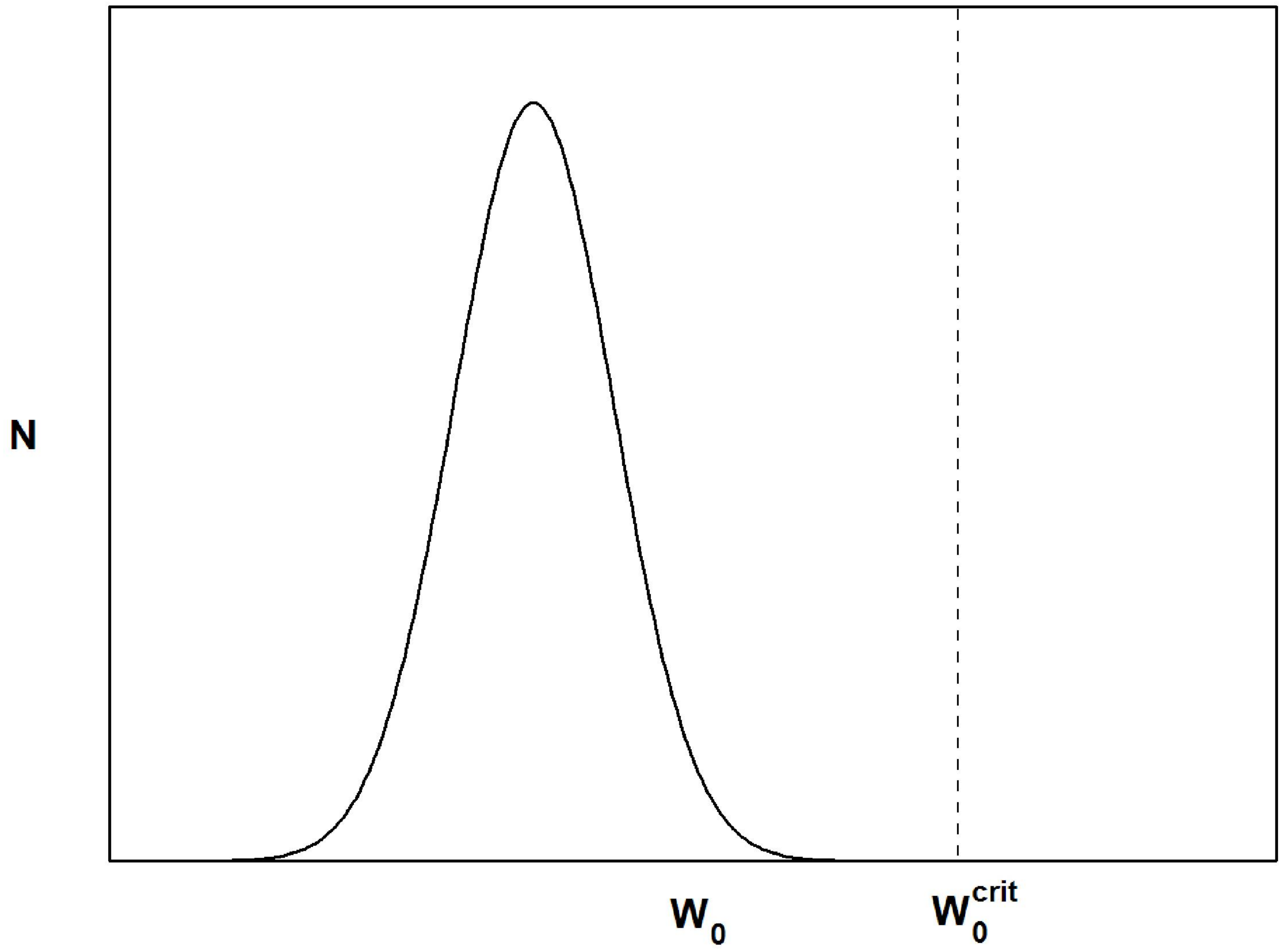

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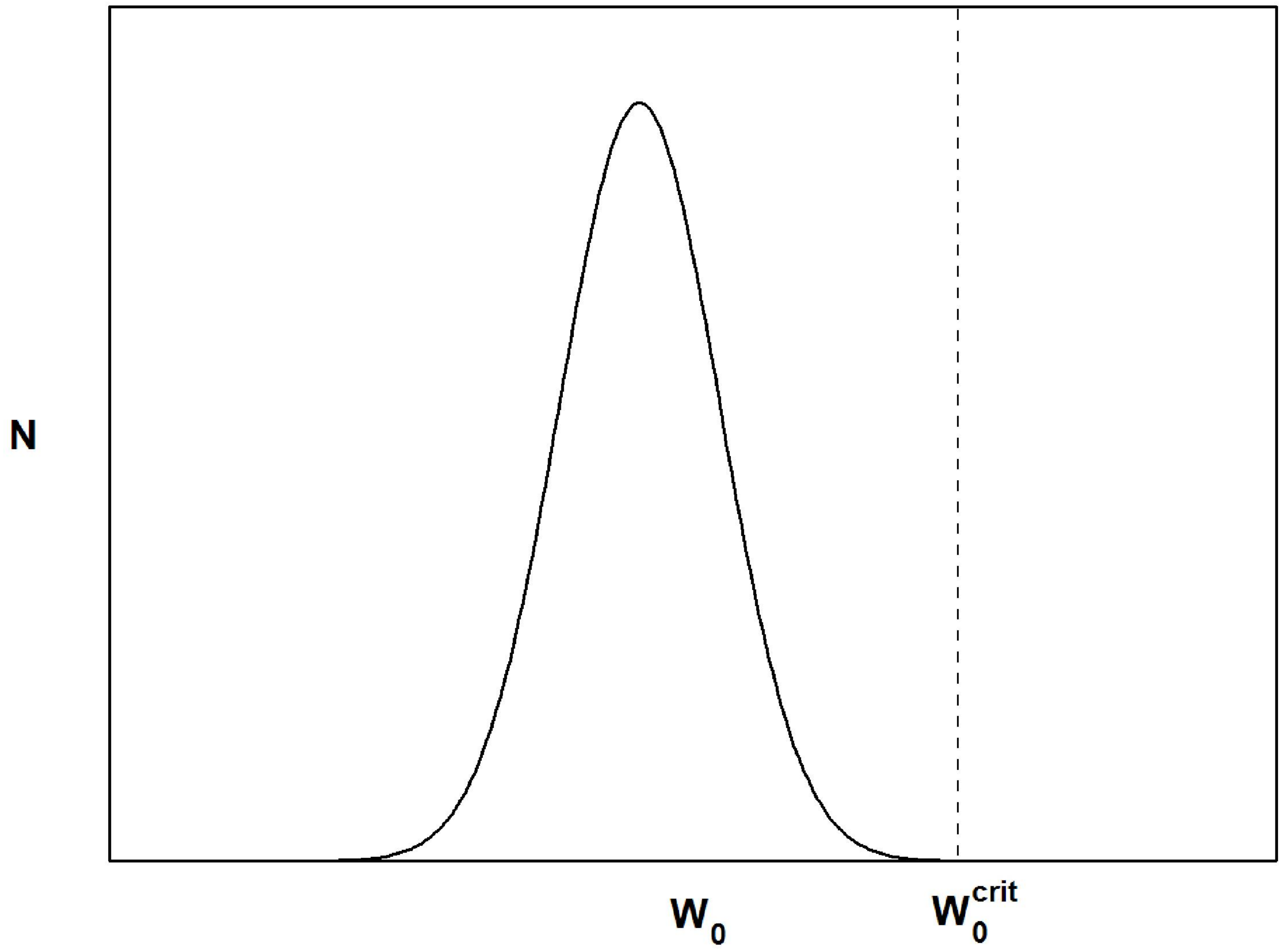

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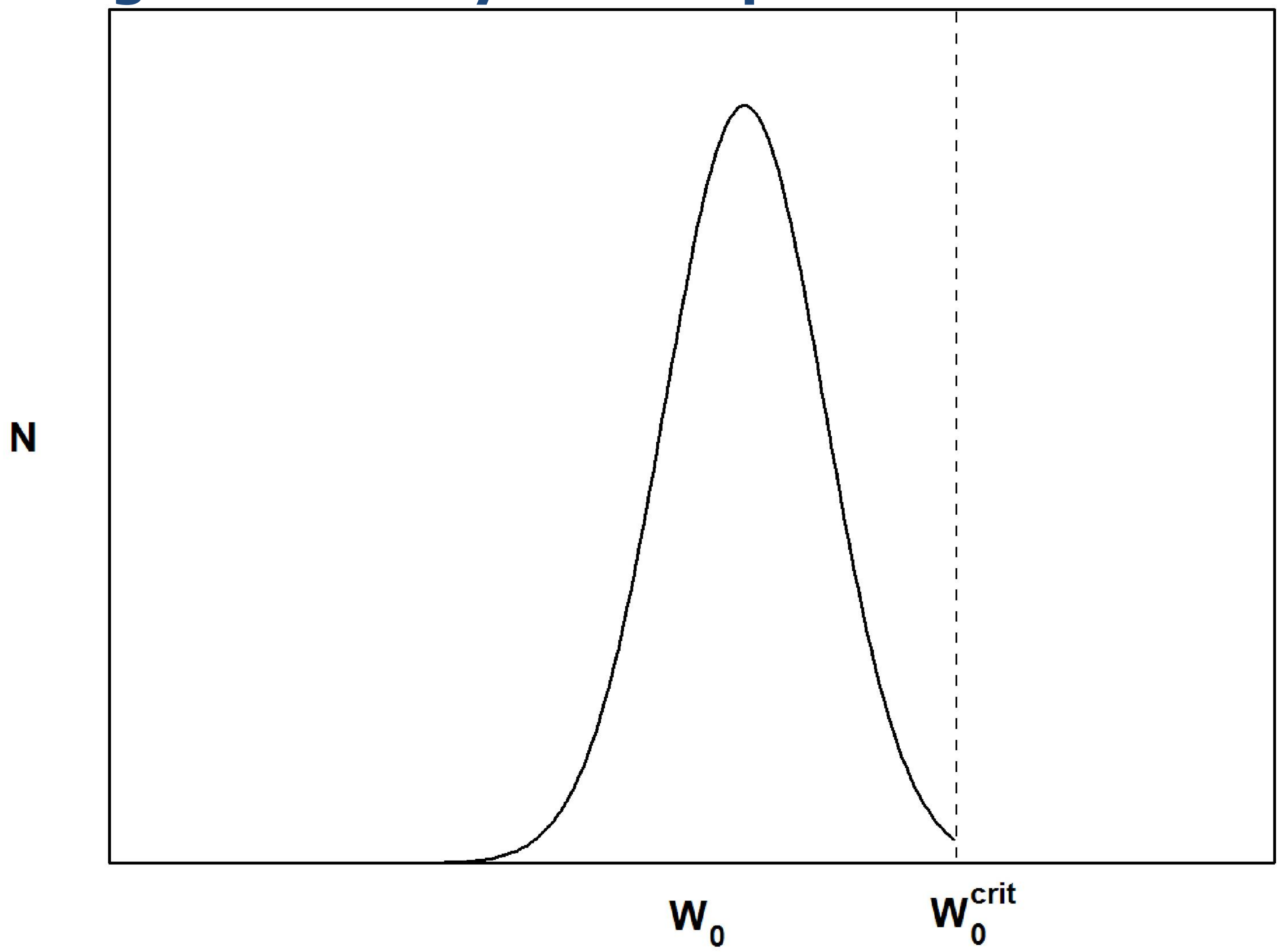
Effects due to gravothermal catastrophe begin to modify the shape of the distribution



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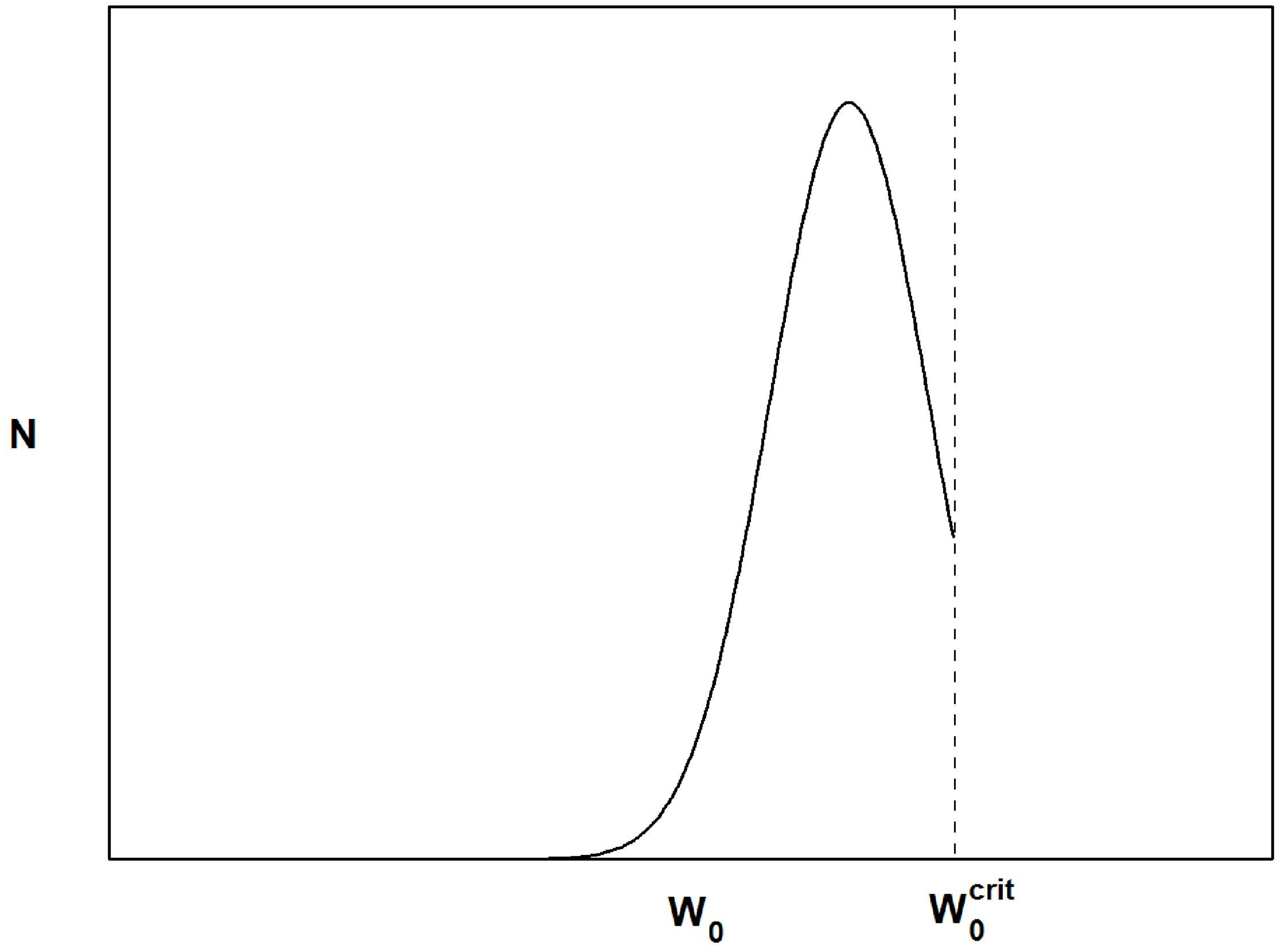

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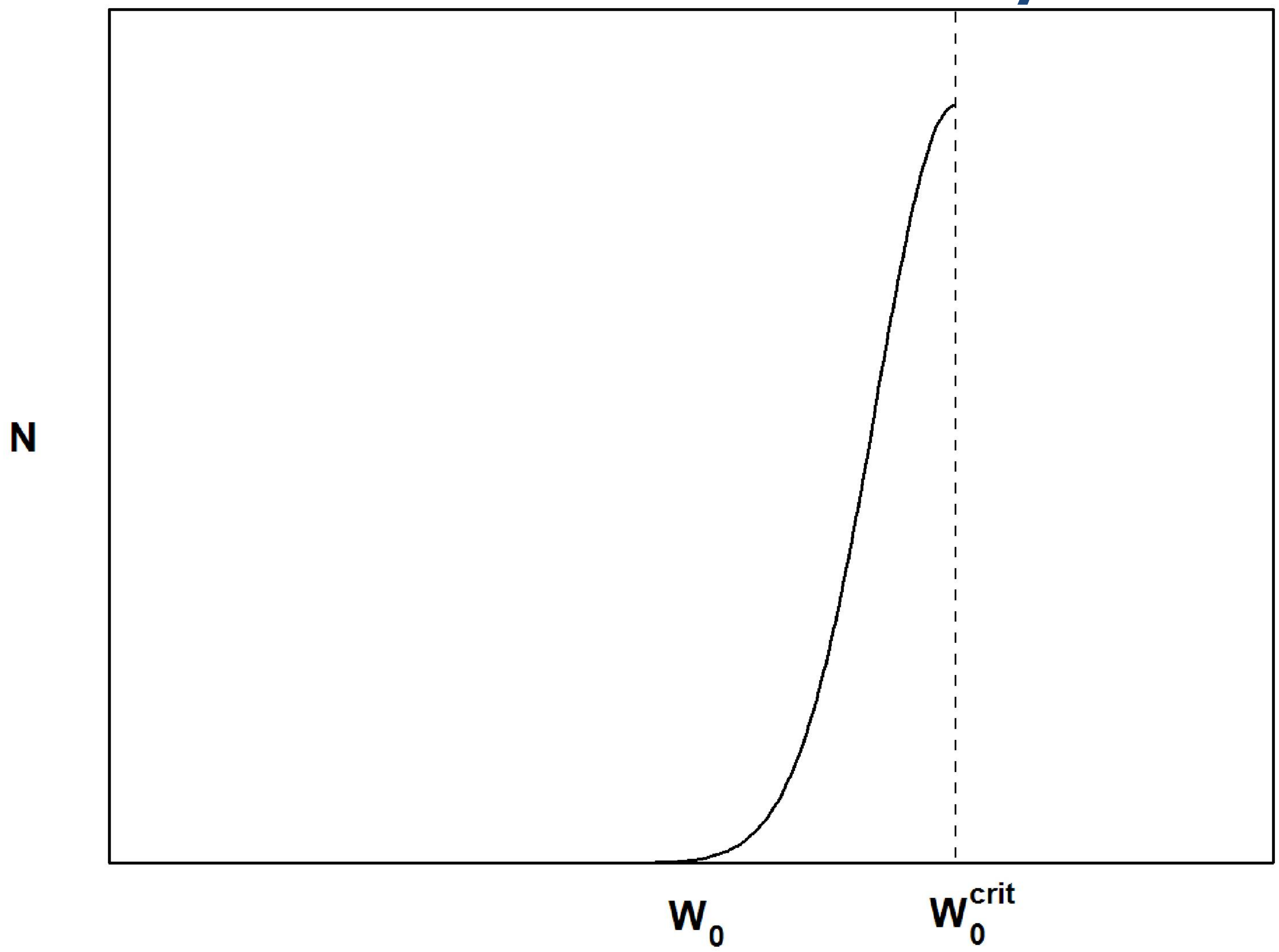
The maximum coincides with the critical point
The distribution has become asymmetric



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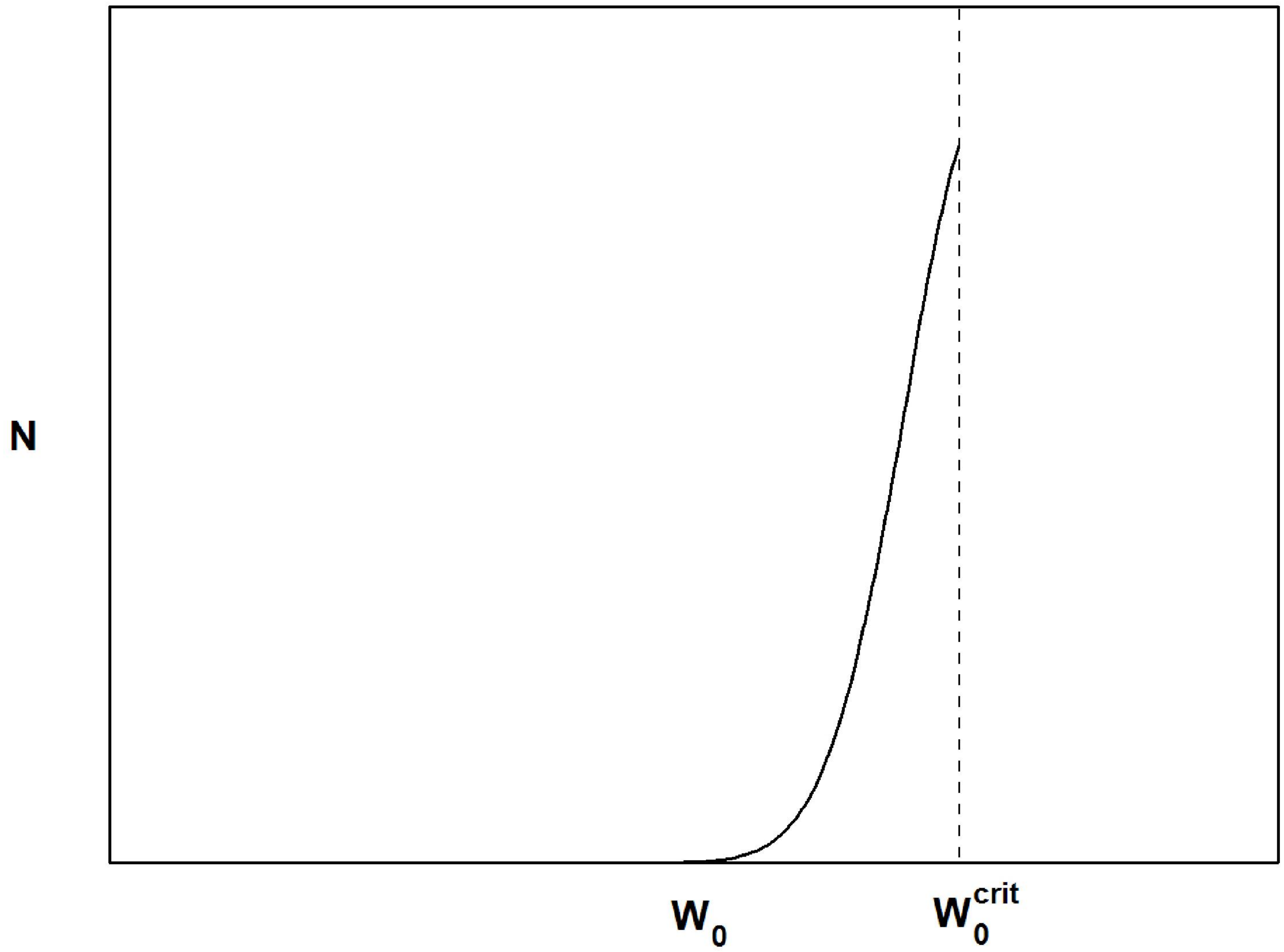

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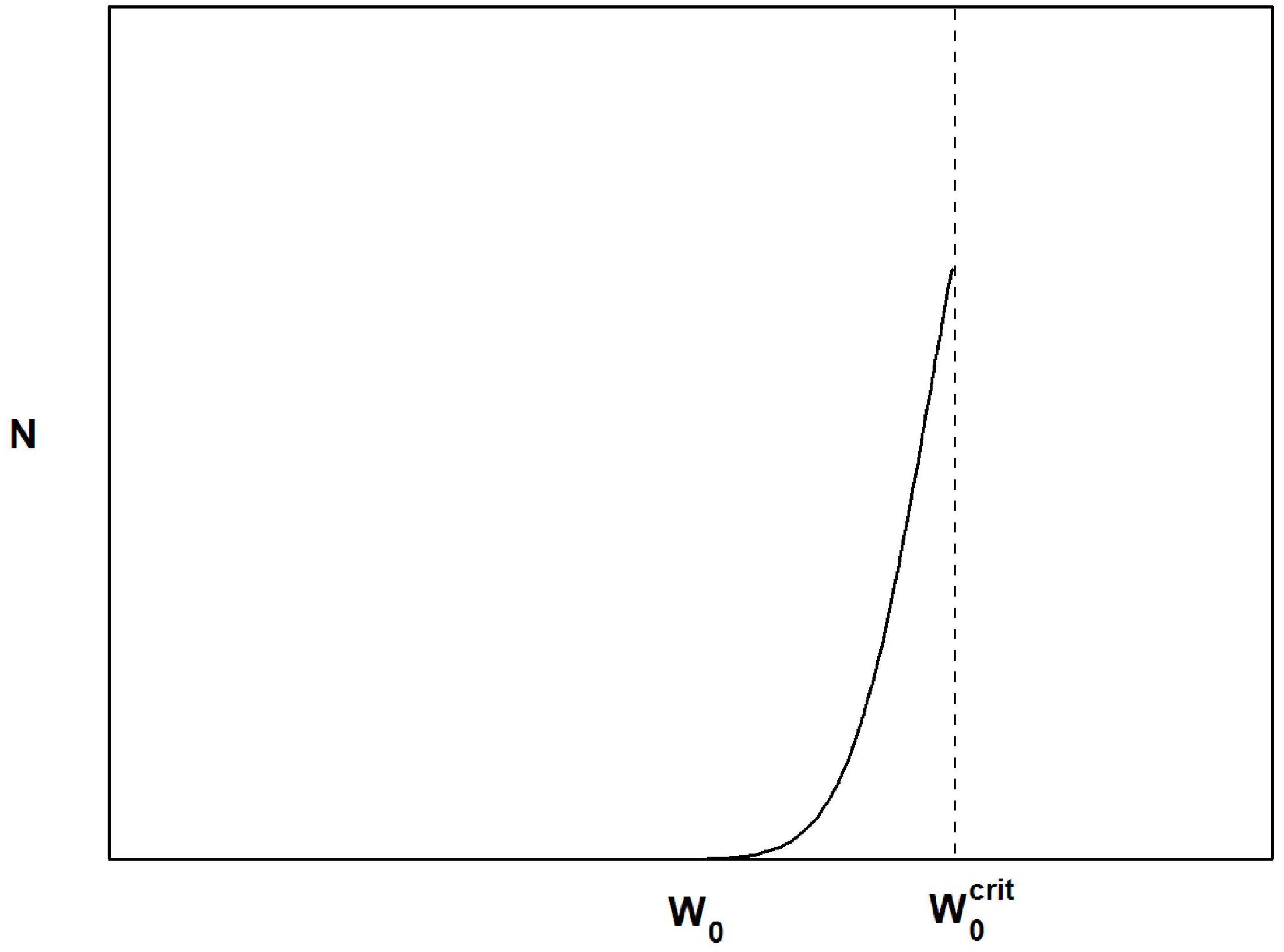

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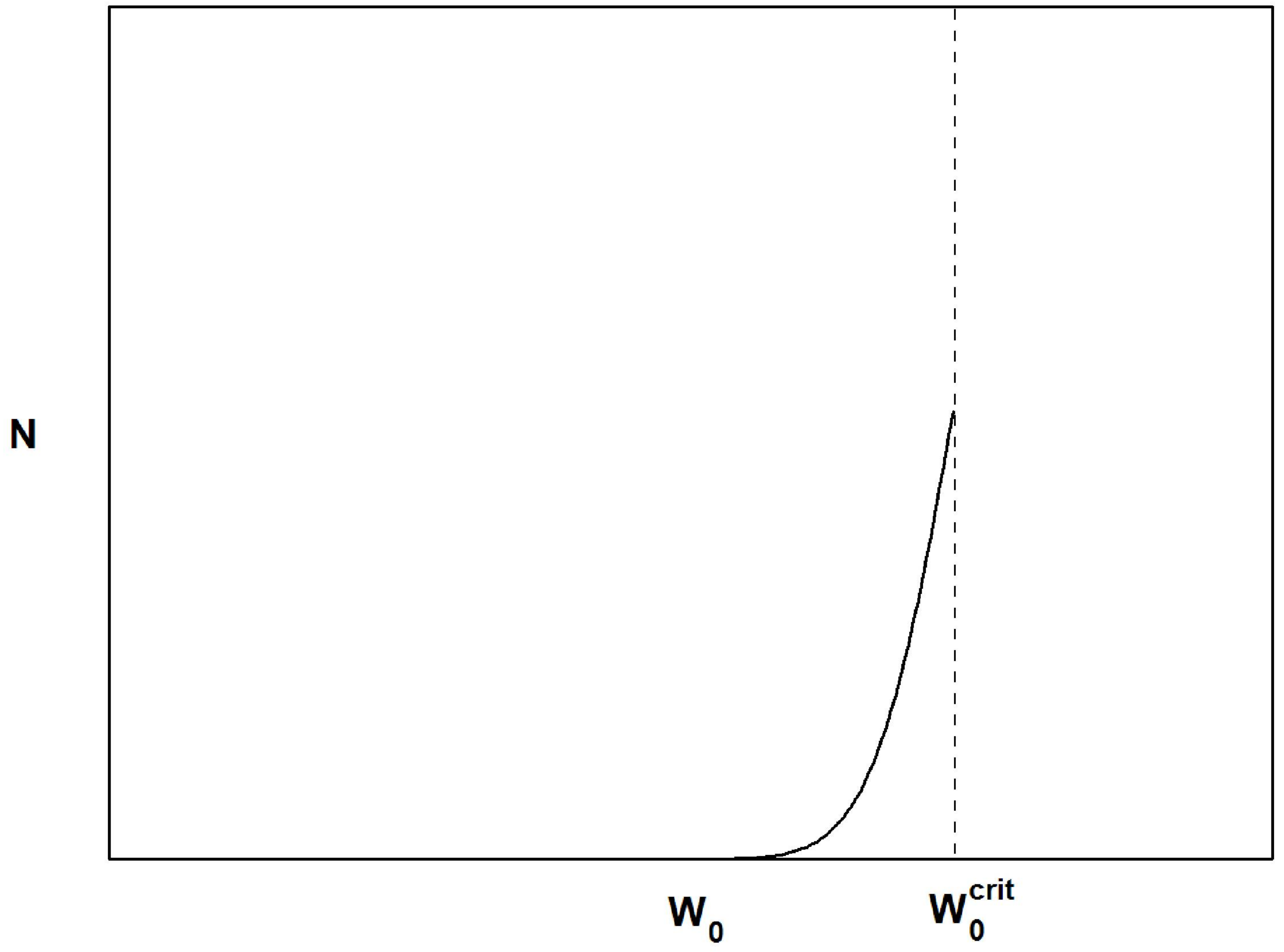

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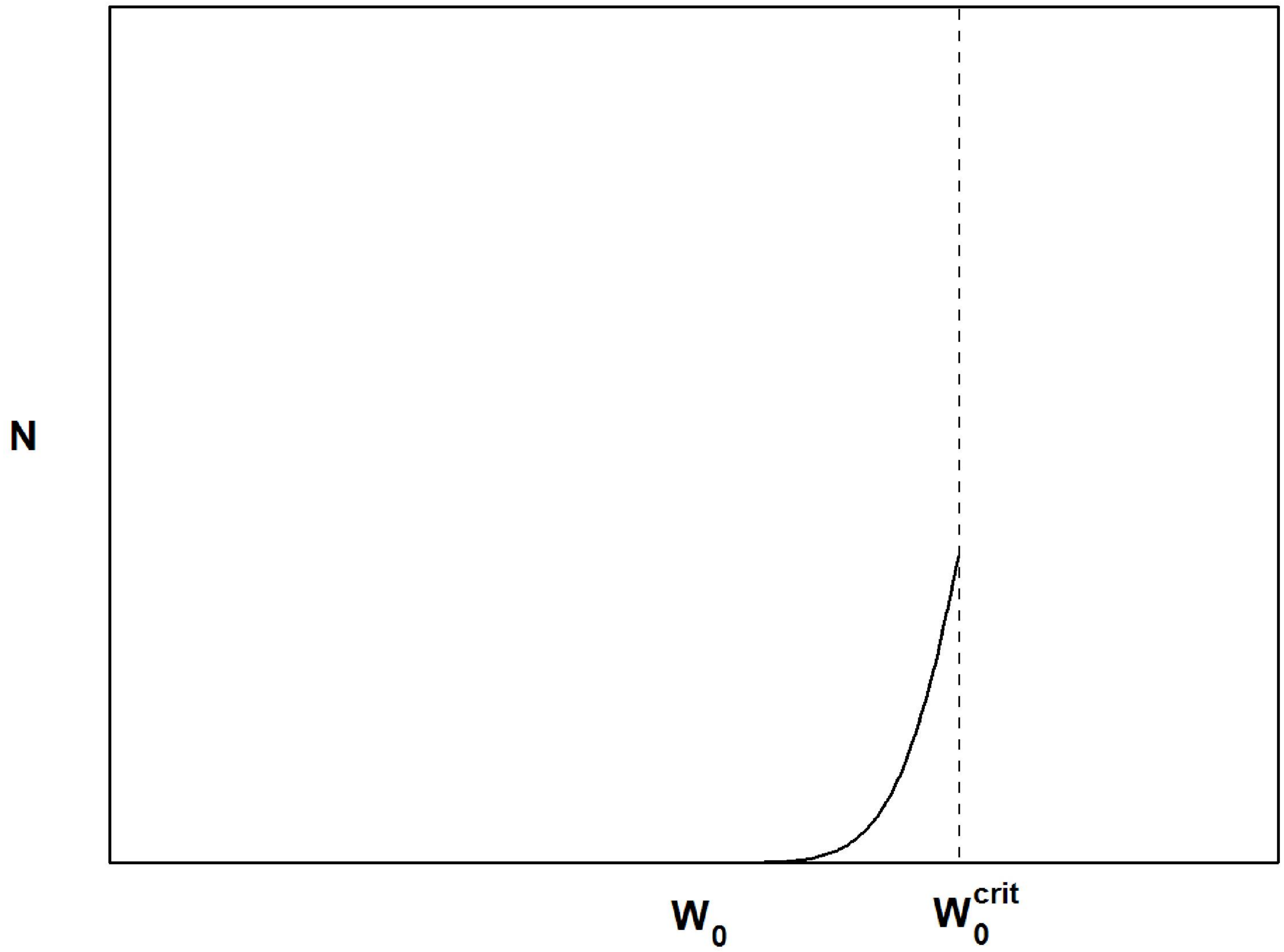

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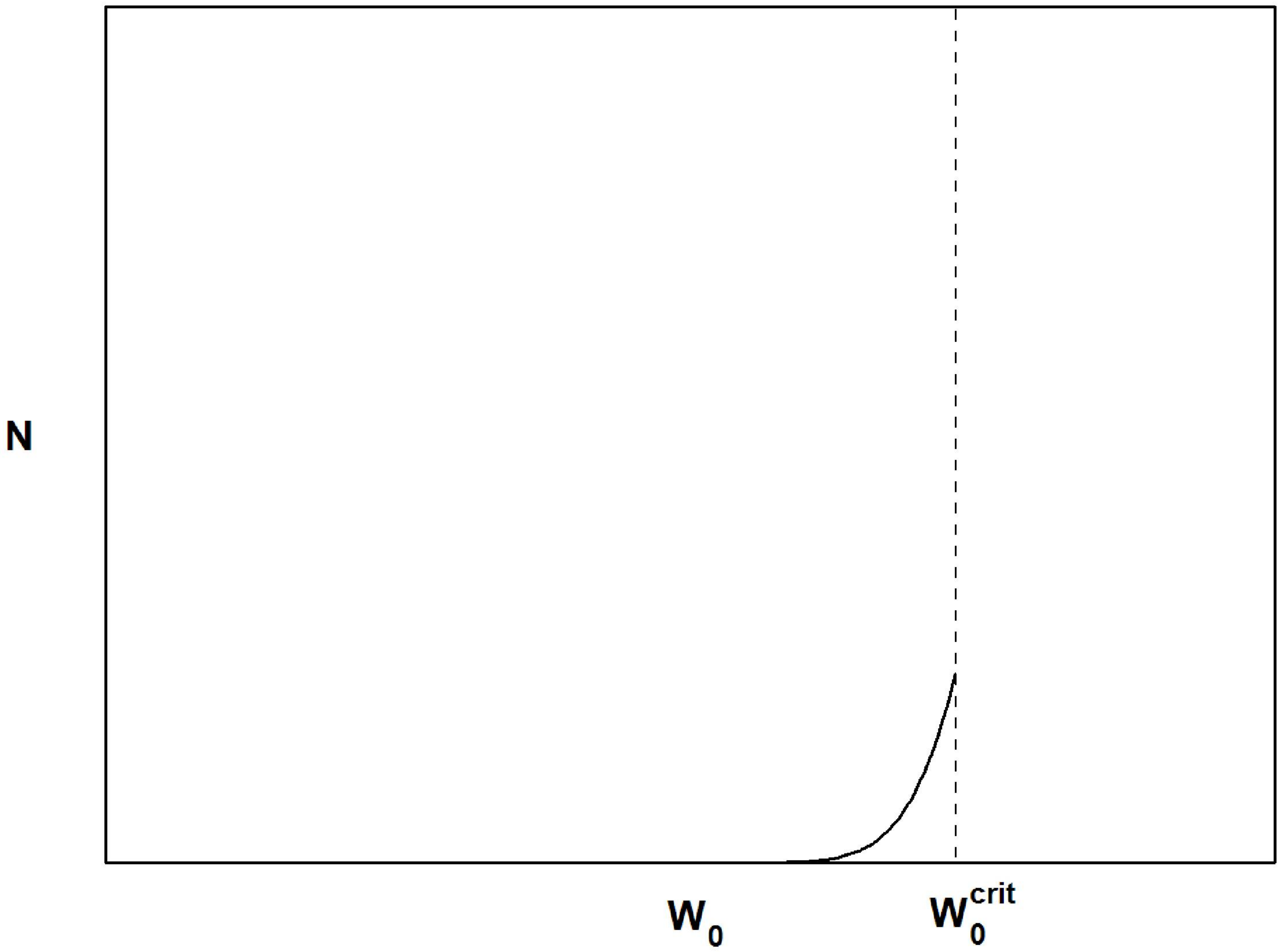

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Consequences on the GCs distribution

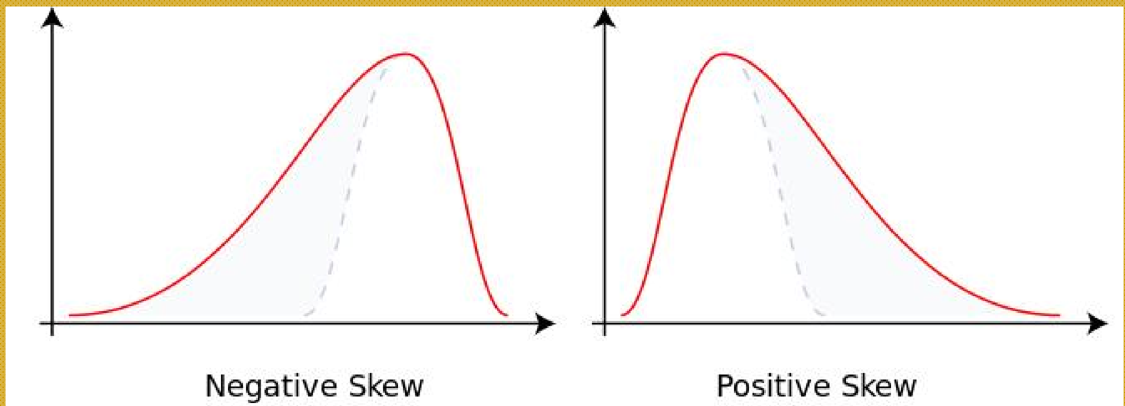
We expect *maximum* of the distribution MUST coincide with critical point

MOREOVER

We expect GCs distribution MUST be asymmetric

Katz "Gaussian assumption" failed

Introducing "Pearson mode skewness" $\alpha_P = \dfrac{\text{mean} - \text{mode}}{\text{standard deviation}}$



We expect a negative skew

Using Peterson K-data (51 clusters):

$$\alpha_P = -0.8648$$

W_0-distribution of MW GCs

New helpful observations (upgrading Peterson data)

Harris Catalogue of MW GCs: 157 objects

(W.E. Harris, 1996 - Latest Edition: 2010)

Excluding:

- 21 PCC clusters

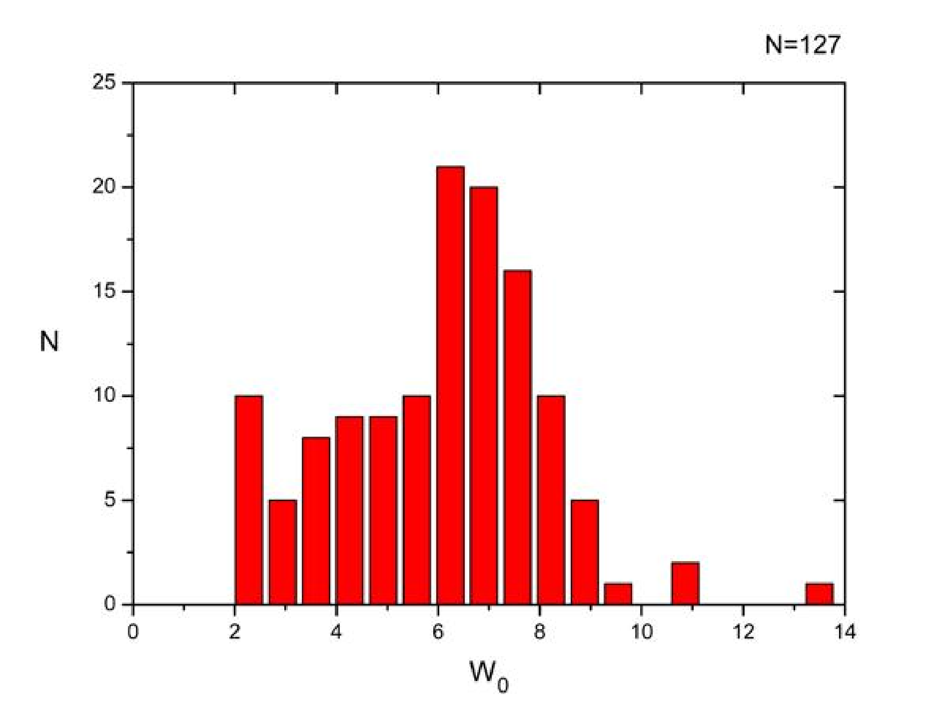
- 8 suspected PCC clusters

- 1 cluster without useful parameters (Pyxis)

→ We have available a sample of 127 clusters

skewness using Harris Catalogue

$$\alpha_P = -0.5097$$



The effects of gravothermal catastrophe are clearly visible: we have an asymmetric distribution of W_0 values

The choice of the distribution function

"Pearson mode skewness" definitions

for a data sample

$$\alpha_P = \frac{\bar{x} - x_+}{s}$$

X_+ is the most frequent value (mode)

for a distribution function

$$\alpha_P = \frac{\mu - x_m}{\sigma}$$

X_m is the maximum of the distribution

Choosing Asymmetric Gaussian distribution function

$$f_L = \frac{A}{\sigma_L (B+1)\sqrt{\pi/2}} e^{-\frac{(x-x_m)^2}{2\sigma_L^2}} \quad \text{for} \quad x \le x_m$$

$$f_R = \frac{A}{\sigma_L (B+1)\sqrt{\pi/2}} e^{-\frac{(x-x_m)^2}{2B^2\sigma_L^2}} \quad \text{for} \quad x \ge x_m$$

where $B = \frac{\sigma_R}{\sigma_L}$

Asymmetric Gaussian skewness

Definitions

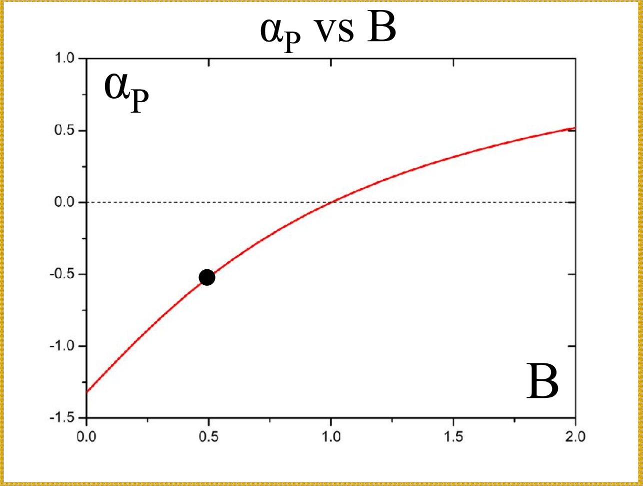
$$\text{normalization}: \quad \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1 \qquad \text{for} \quad A = 1$$

$$\Rightarrow \quad \mu = \int_{-\infty}^{+\infty} x\, f(x)\,\mathrm{d}x\,; \quad \sigma^2 = \int_{-\infty}^{+\infty} x^2 f(x)\,\mathrm{d}x - \left[\int_{-\infty}^{+\infty} x\, f(x)\,\mathrm{d}x\right]^2$$

Skewness

$$\mu = x_m + \sqrt{\frac{2}{\pi}}(B-1)\sigma_L\,; \quad \sigma^2 = \frac{B^3+1}{B+1}\sigma_L^2 - \frac{2}{\pi}(B-1)^2\sigma_L^2$$

$$\alpha_P = \frac{B-1}{\sqrt{\frac{\pi}{2}\frac{B^3+1}{B+1} - (B-1)^2}} \quad \Rightarrow \quad \alpha_P = -\,0.5097 \quad B = 0.5071$$



Fitting data from W_0-distribution

Best binning

- ***sample dimension***: N=127 clusters
- ***range of W_0 values***: 1.97÷13.22

$$\sqrt{N} \le n_{bin} \le 2\sqrt{N}$$

$$d_{bin} = range / n_{bin}$$

we perform fit on histograms with different value of d_{bin}

$$0.6 \le d_{bin} \le 0.9$$

Our choice

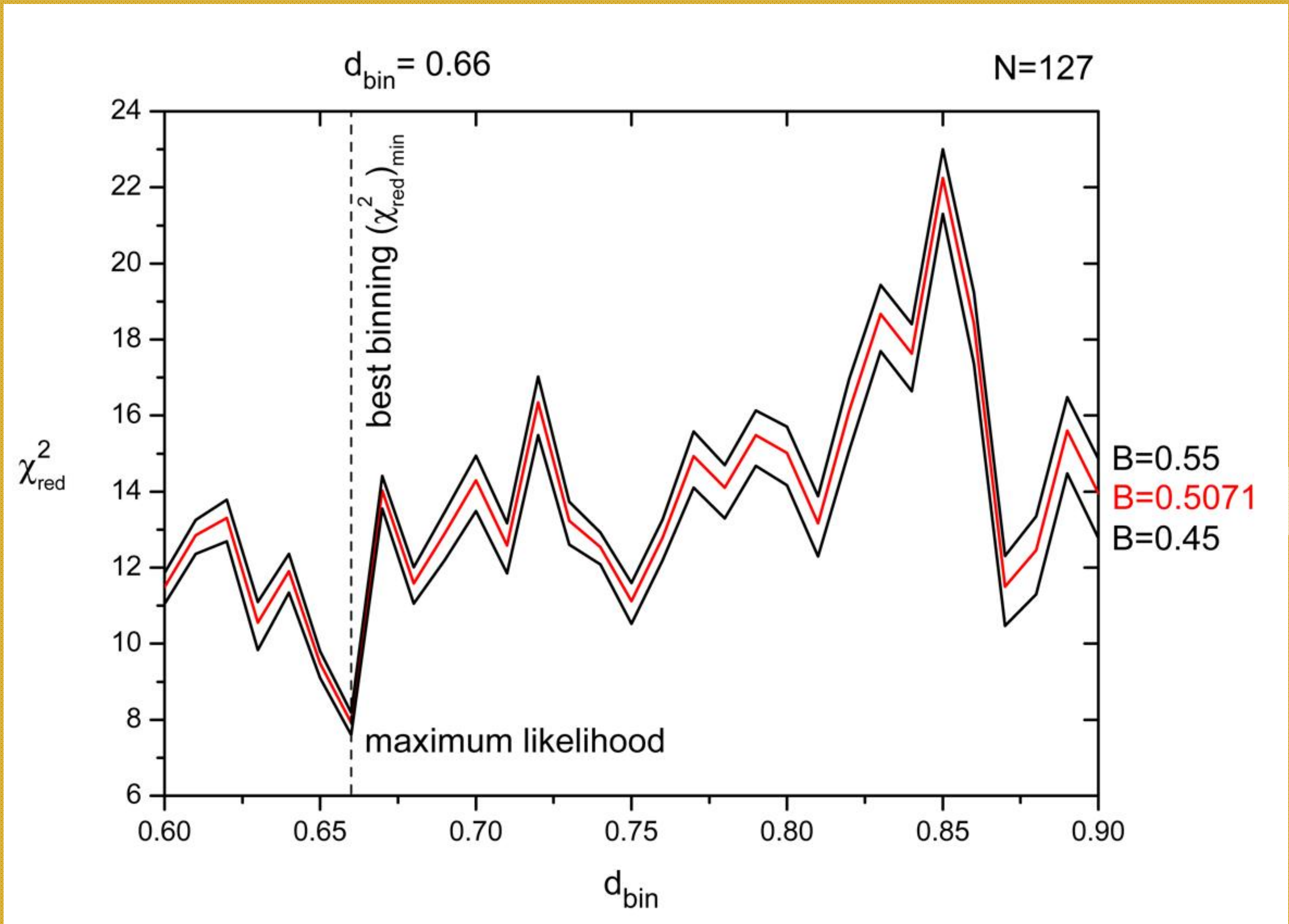
Best fit

we perform fit, fixing B in a range compatible with skewness of data distribution ($\alpha_P \sim 0.5071$)

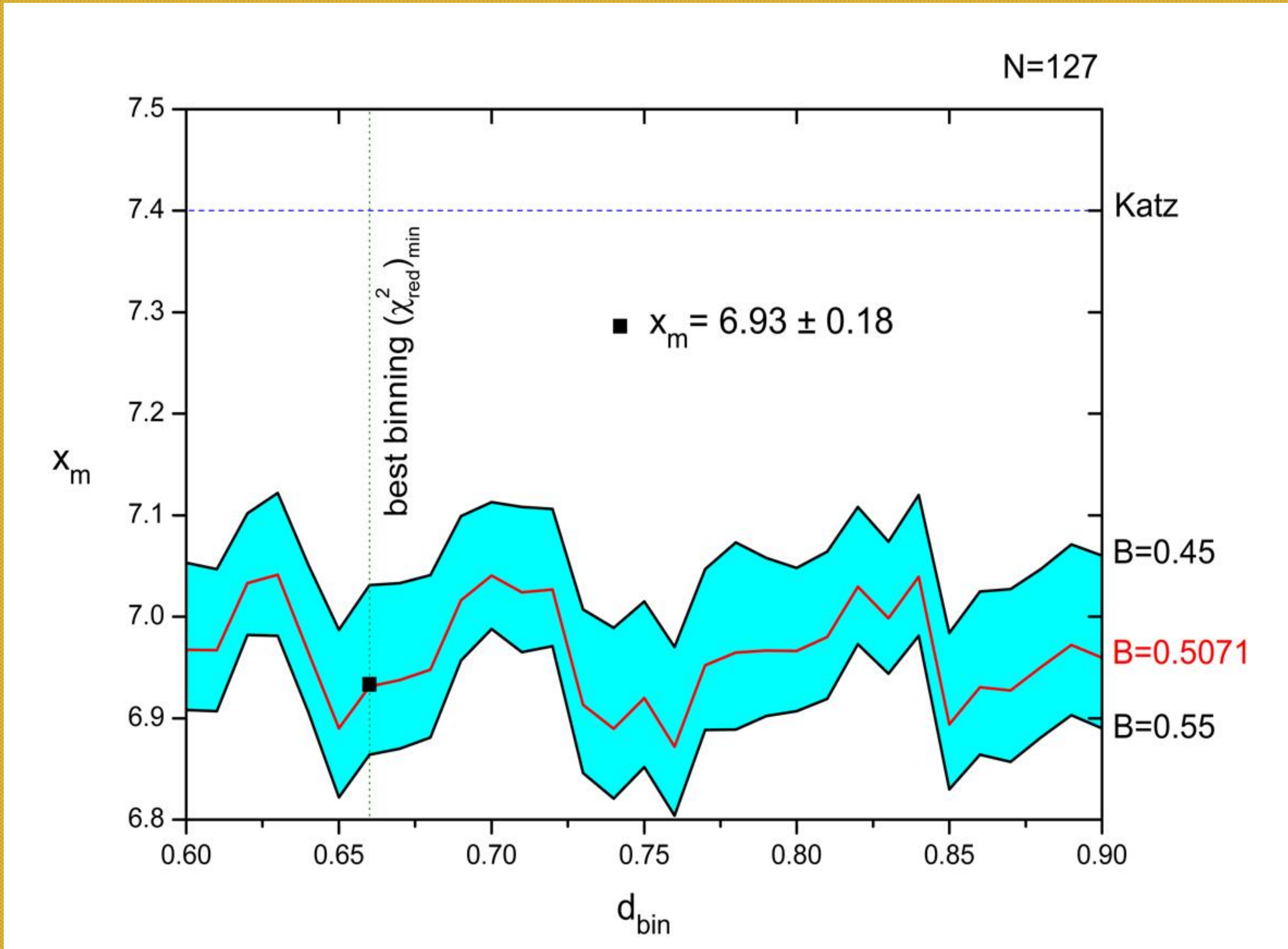
$$0.45 \le B \le 0.55$$

Our choice

Searching for maximum likelihood binning



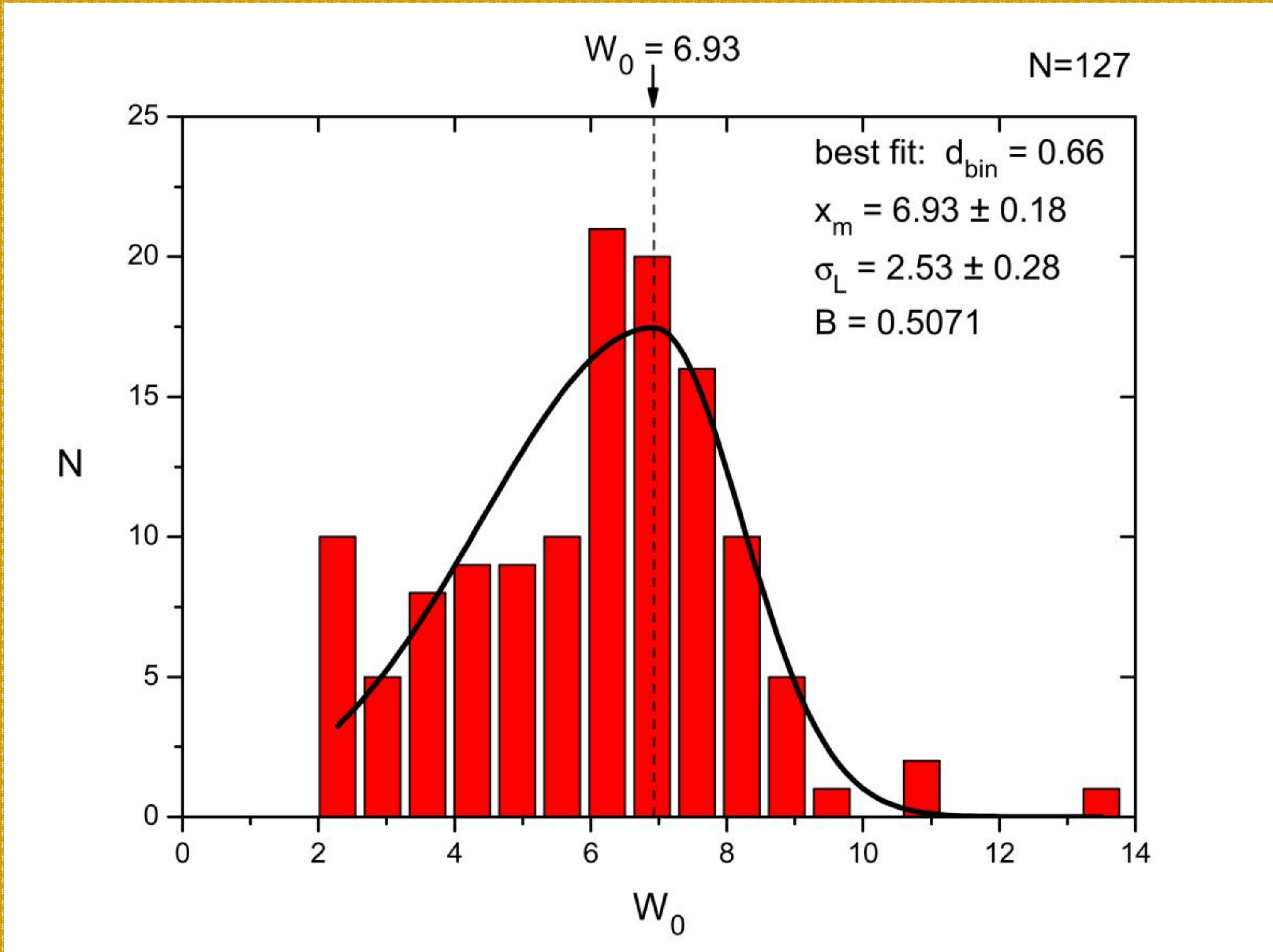
Searching for maximum of W_0-distribution



Does W_0=7.4 actually represent the onset of gravothermal catastrophe?

The answer is clearly NO. The maximum of W_0-distribution is far away from coinciding with critical value !

Best fit of MW GCs distribution



In any case maximum cannot be in correspondence of W_0=7.4

A new strategy?

- Observational data say us there is a problem with the onset of gravothermal instability, in order to obtain the coincidence between maximum of W_0-distribution and critical value.
- The gap between the current stability limit $W_0 = 7.4$ and the peak value of the distribution at $W_0 = 6.93$ is significantly large, and only a W_0-distribution with a skewness completely different from observational data ($B \ll 1$) could justify such a stability limit.

THEN

We must reconsider the calculation of the Fokker-Planck solution focusing our attention on the definition of the distribution function and taking into account that we need a thermodynamic equilibrium distribution which maximizes the entropy. The mandatory choice is considering a Boltzmann form of the distribution and solving with respect to the Hamiltonian.

Restarting from basic principles

We are aware of the steps forward in the understanding
the dynamics (and not only) of globular clusters

BUT

forget for a moment models taking into account multiple effects

In this theoretical analysis we do not consider (but in future we can)

- multi-mass models and their effect in the dynamical evolution of GCs;
- binary stars formation, possible presence of BH, BSS and so on;
- effects of stellar evolution in dynamics of GCs;
- effects of possible anisotropy or rotation in a GC.

→ only a **single-mass model** with the velocity distribution function arising from Fokker-Planck equation, in order to avoid to walk too often the road of phenomenology! THIS IS A NEW STARTING POINT

"Even the most beautiful penthouse is in danger of collapse
if the building does not rest on a solid foundation"

Fokker-Planck equation

Spitzer-Harm form of the Fokker-Planck equation (1958)

$$\frac{d}{dx}\left[G(x)\left(\frac{dg(x)}{dx}+2xg(x)\right)\right]+\lambda x^2 g(x)=0$$

where

$$G(x)=\frac{4}{\sqrt{\pi}x}\int_0^x e^{-y^2}y^2\,dy$$

$$x = v/\sqrt{2}\sigma \text{ and } \sigma = \sqrt{k\theta/m}$$

Distribution function f

$$f(t,x)=e^{-\lambda t/t_R}g(x)$$

λ is the fractional loss rate of the stellar evaporation

t_R is the **reference (relaxation) time**

$$g(x)=Ae^{-H/k\theta} \quad ; \quad H=H(x)=H_0(x)+H_1(x)$$

Solution of the Fokker-Planck equation

$$g'(x)+2xg(x)=-\frac{\lambda}{G(x)}\int_0^x g(y)y^2\mathrm{d}y;\quad g(x)=Ae^{-H/k\theta};\quad H(x)=H_0(x)+H_1(x,\lambda)$$

$$g(0)=A;\quad g'(0)=0;\quad g(x_e)=0\qquad H(0)=0;\quad H(x_e)=\infty;\quad H_1(x,0)=0$$

Inserting g(x) in equation and matching terms with same power of λ

$$H_0(x)=k\theta x^2;\quad \mathrm{e}^{-H_1(x,\lambda)/k\theta}\cong 1-\frac{\lambda}{k\theta}\left.\frac{\partial H_1}{\partial\lambda}\right|_{\lambda=0}$$

$$\left.\frac{\partial H_1}{\partial\lambda}\right|_{\lambda=0}=\frac{\sqrt{\pi}}{8}k\theta(\mathrm{e}^{x^2}-1);\quad H_1(x,\lambda)=-k\theta\ln\left[1-\frac{\sqrt{\pi}}{8}\lambda(\mathrm{e}^{x^2}-1)\right]$$

The cutoff condition $H(x_e)=\infty$ implies that $\lambda=\frac{8}{\sqrt{\pi}}(\mathrm{e}^{x_e^2}-1)^{-1}$ and

$$H(x)=k\theta\left\{x^2-\ln\left[1-\frac{\sqrt{\pi}}{8}\lambda(\mathrm{e}^{x^2}-1)\right]\right\}$$

"Blessed are those who have not seen and have believed!" (John 20, 29)

H-distribution function vs King one

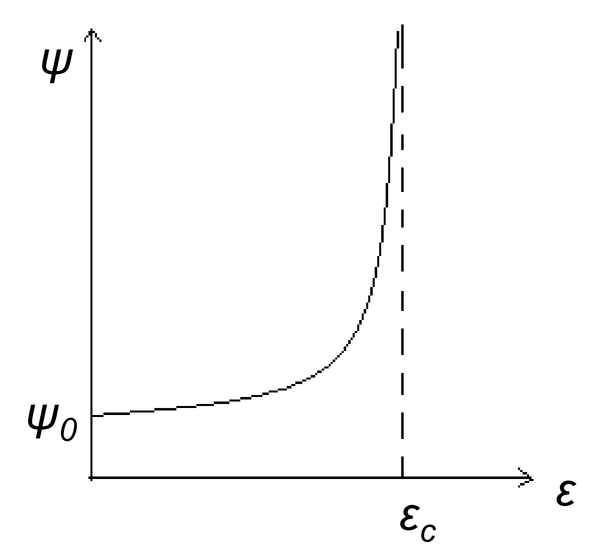
Comparing the H-distribution function containing the effective potential ψ with old King form

$$f(\varepsilon) = Be^{-H/k\theta}$$

$$H = \varepsilon + m\varphi + m\psi$$

$$m\psi = -k\theta \ln\left(1 - e^{(\varepsilon-\varepsilon_c)/k\theta}\right)$$

$$\Rightarrow \begin{cases} f_K(\varepsilon) = A\,(e^{-\varepsilon/k\theta} - e^{-\varepsilon_c/k\theta}) & for\ \varepsilon \le \varepsilon_c \\ f_K(\varepsilon) = 0 & for\ \varepsilon > \varepsilon_c \end{cases}$$



thermodynamic quantities

$$N = AV\int_0^{\varepsilon_c} f\sqrt{\varepsilon}\,d\varepsilon; \quad \prod = A(k\theta)\int_0^{\varepsilon_c} f\sqrt{\varepsilon}\,d\varepsilon$$

$$U = AV\int_0^{\varepsilon_c} fH\sqrt{\varepsilon}\,d\varepsilon; \quad S = AVk\int_0^{\varepsilon_c} f[1-\ln(f)]\sqrt{\varepsilon}\,d\varepsilon$$

The exponential form of *f* allows to properly define entropy

We expect to find this form of potential in addition to the gravitational contribution in GCs

A new formulation

- *Old formulation leads to miss an important term (effective potential) in the expression of total energy which is not affecting kinematics of the stars but is crucial in the analysis of gravothermal instabilities, being responsible of shrinking of volume of the available phase space.*

- *Since 50 years, this old formulation has addressed all the works on dynamics of globular clusters to a phenomenologic analysis not rigorously collisional, as we could expect, in order to explain multiple effects affecting GCs.*

- *The correct way is inserting the variables in Fokker-Planck equation and solve it, even numerically if necessary, in order to obtain the correct distribution taking into account these effects.*

Star orbits (not affected)

$\Omega = H/m\sigma^2$ (total energy)

-constant of motion-

$\Delta = \psi/\sigma^2$ (effective potential)

-constant of motion-

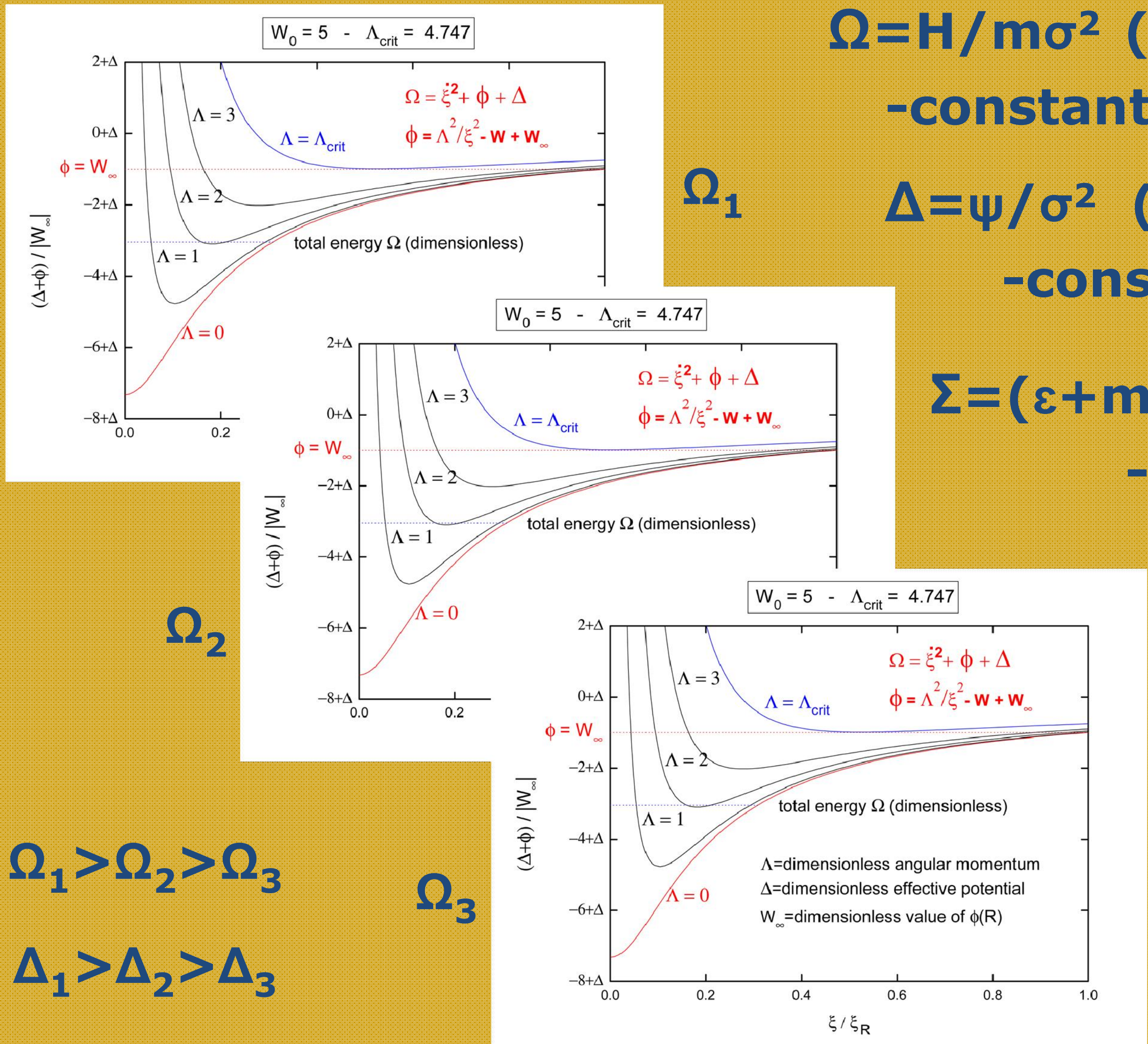
$\Sigma = (\varepsilon + m\phi)/m\sigma^2$ (kin+grav energy)

-constant of motion-

Ω_1

Ω_2

Ω_3



$\Omega_1 > \Omega_2 > \Omega_3$

$\Delta_1 > \Delta_2 > \Delta_3$

Lagrangian formalism

☑ **checked**

Hamiltonian formalism

☑ **checked**

Thermodynamics: a dual theory

First Law

$$dU = \theta dS - \Pi dV + \alpha dN + Nm\left\langle d\psi \right\rangle$$

$$dU_{kin} = TdS - PdV + \langle \mu_0 \rangle dN + N(d\langle \mu_0 \rangle - \langle d\mu_0 \rangle)$$

Eulero's relations

$$U = \theta S - \Pi V + \alpha N$$

$$U_{kin} = TS - PV + \langle \mu_0 \rangle N$$

Gibbs-Duhen relations

$$Nm\langle d\psi \rangle = Sd\theta - Vd\Pi + Nd\alpha$$

$$N(d\langle \mu_0 \rangle - \langle d\mu_0 \rangle) = SdT - VdP + Nd\langle \mu_0 \rangle$$

Pressure and temperature

kinetic pressure

$$P = \frac{1}{3} A \int f\, q \frac{d\varepsilon}{dq} d^3 q$$

thermodynamic pressure

$$\Pi = \frac{1}{3} A \int f\, q \frac{dH}{dq} d^3 q$$

$$\longrightarrow \quad \Pi = P + \frac{N}{3V} \left\langle q \frac{\partial \psi}{\partial q} \right\rangle$$

kinetic temperature

$$kT = \left\langle q_i \frac{\partial \varepsilon}{\partial q_i} \right\rangle$$

thermodynamic temperature

$$k\theta = \left\langle q_i \frac{\partial H}{\partial q_i} \right\rangle$$

$$\longrightarrow \quad k\theta = kT + \frac{1}{3} \left\langle q \frac{\partial \psi}{\partial q} \right\rangle$$

Relation between kinetic temperature and mean velocity of stars

$3kT = m \langle v^2 \rangle$

⇨

kinetic equation of state : $PV = NkT$

thermodynamic equation of state : $\Pi V = Nk\theta$

Thermodynamic Equilibria

Thermal equilibrium

$$dS_{tot} = 0; \quad N, V\ const \quad \Rightarrow \quad \delta\theta = 0$$

constant T-temperature

(K-temperature is not constant all over the equilibrium configuration)

Mechanical equilibrium

$$dS_{tot} = 0; \quad N, S\ const \quad \Rightarrow \quad \delta\Pi + \frac{N}{V}[< \delta\psi > + m\delta\varphi] = 0$$

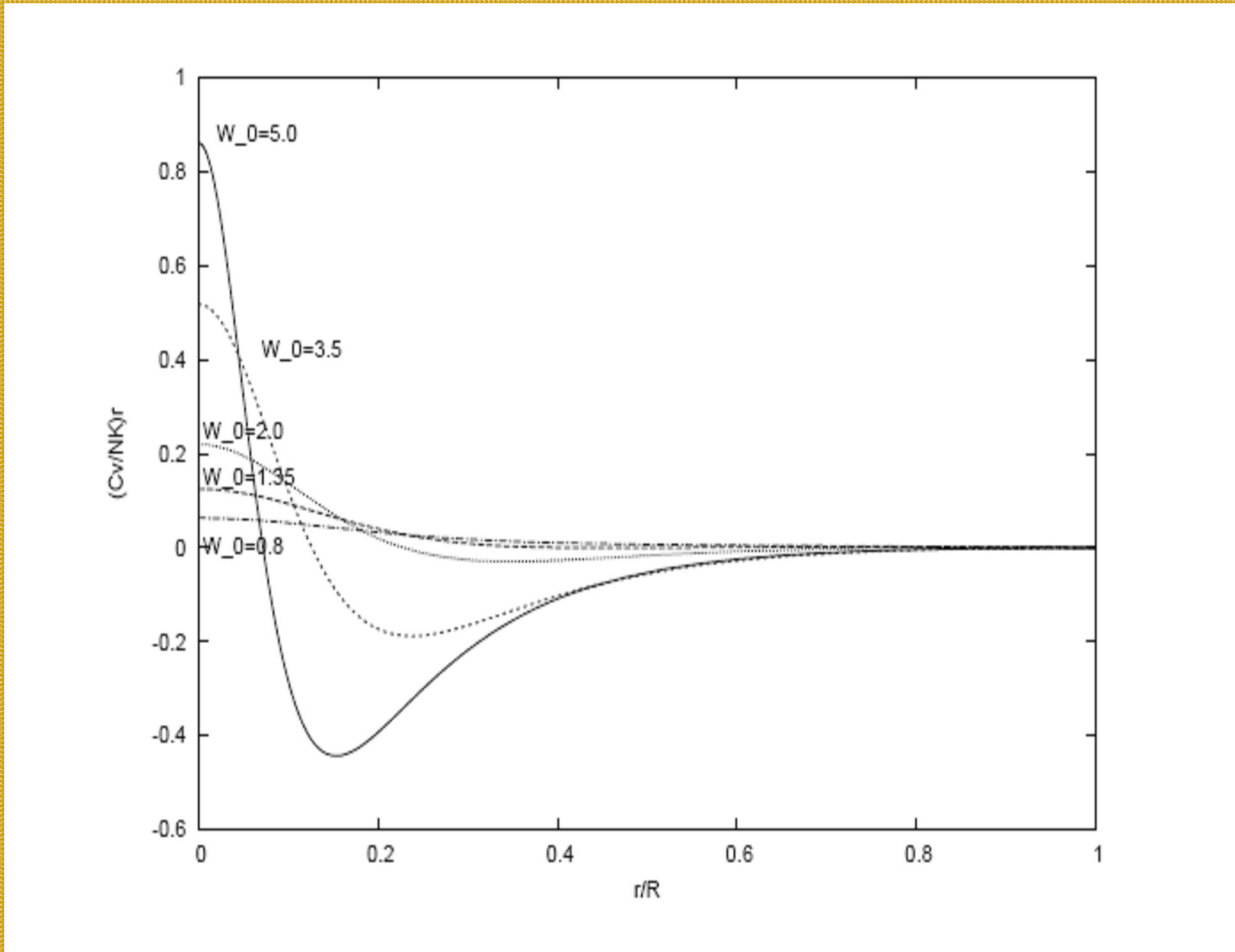
hydrostatic equilibrium $\quad \delta P + \rho\delta\varphi = 0$

Chemical equilibrium

$$dS_{tot} = 0; \quad S, V\ const \quad \Rightarrow \quad \delta\alpha = 0$$

constant T-chemical potential $\quad \delta\alpha_0 + m\delta\varphi = 0$

Specific heat profiles



$$dU = dQ - \Pi dV + Nm\langle d\psi \rangle$$

$$C_V = \left.\frac{dQ}{d\theta}\right|_V = \frac{dU}{d\theta} - Nm\left\langle \frac{d\psi}{d\theta} \right\rangle$$

Energy U is containing the gravitational energy contribution mφ

behaviour of the specific heat **C_v / Nk** in function of the radial coordinate r for different values of **W_0** (regions with negative **C_v** for **W_0 >1.35**)

Virial theorem & Effective potential

Energy

$$U = A\int_0^{\varepsilon_c} f H \sqrt{\varepsilon}\, d\varepsilon \quad \text{(energy density in a single shell)}$$

$$E_{tot} = E_K + E_{gr} + E_{eff}$$

Kinetic + Gravitational + Effective

Virial Theorem

$$2E_K + E_{gr} = 0$$

⇒

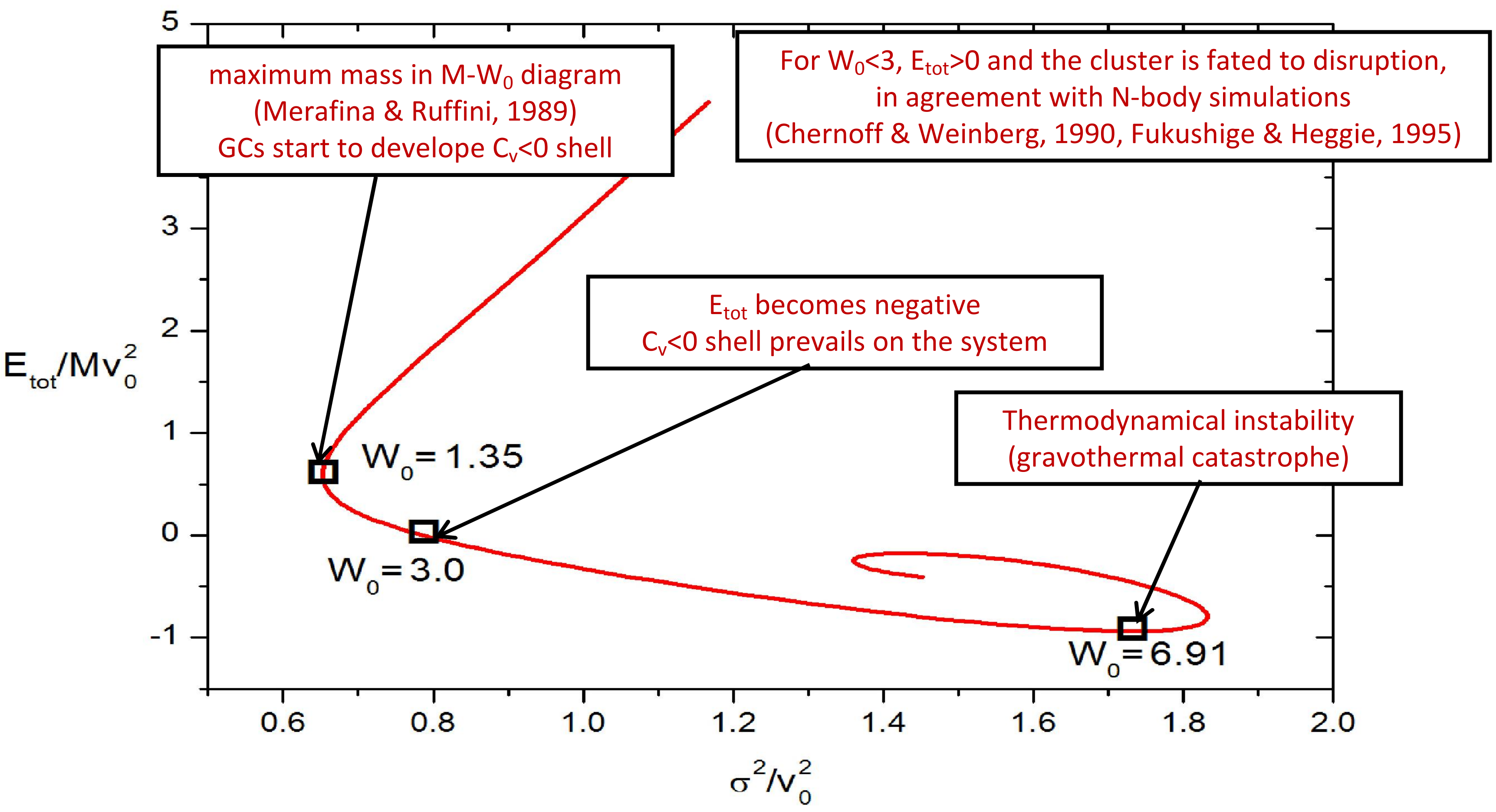
Total energy

$$E_{tot} = -E_K + E_{eff}$$

This additional term implies a different caloric curve

Caloric curve
gravothermal catastrophe onsets earlier

V_0=constant of normalization; $\sigma^2=k\theta/m$



W_0^{crit} ~ W_0^{max} (MW GCs distribution) ⇨ the discrepancy is solved

Modelling a King globular cluster by N-body simulations

Model 1

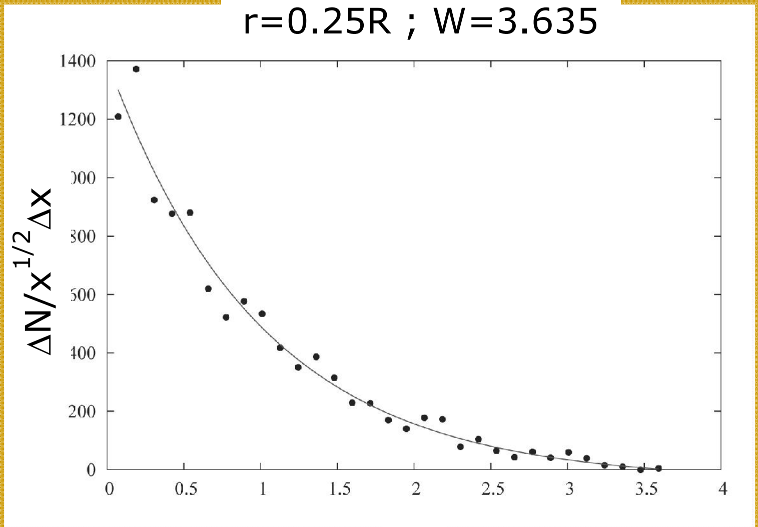
number of stars: $N = 32768$
star mass: $m = 1\ M_{\odot}$
cluster mass: $M = 3.28 \times 10^4\ M_{\odot}$
core radius: $r_c = 10$ pc
$W_0 = 5$

Model 2

number of stars: $N = 262144$
star mass: $m = 1\ M_{\odot}$
cluster mass: $M = 2.62 \times 10^5\ M_{\odot}$
core radius: $r_c = 10$ pc
$W_0 = 5$

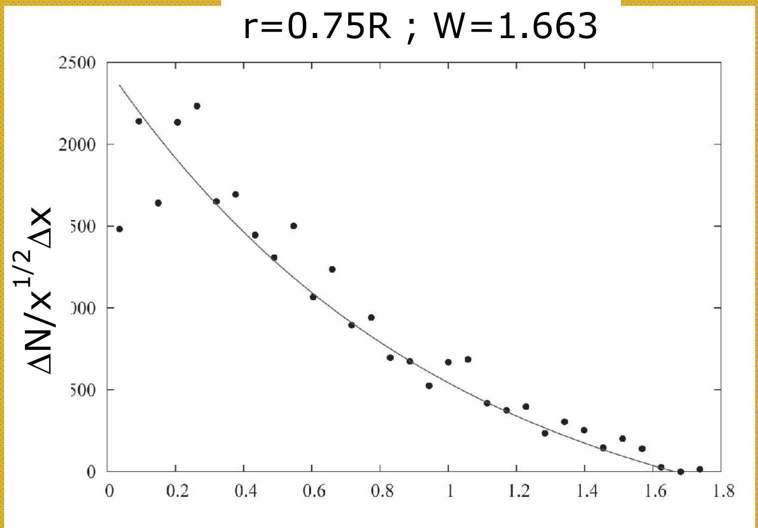
Energy distribution in a single shell: the example of a cluster with $r_c = 2$ pc

r=0.25R ; W=3.635

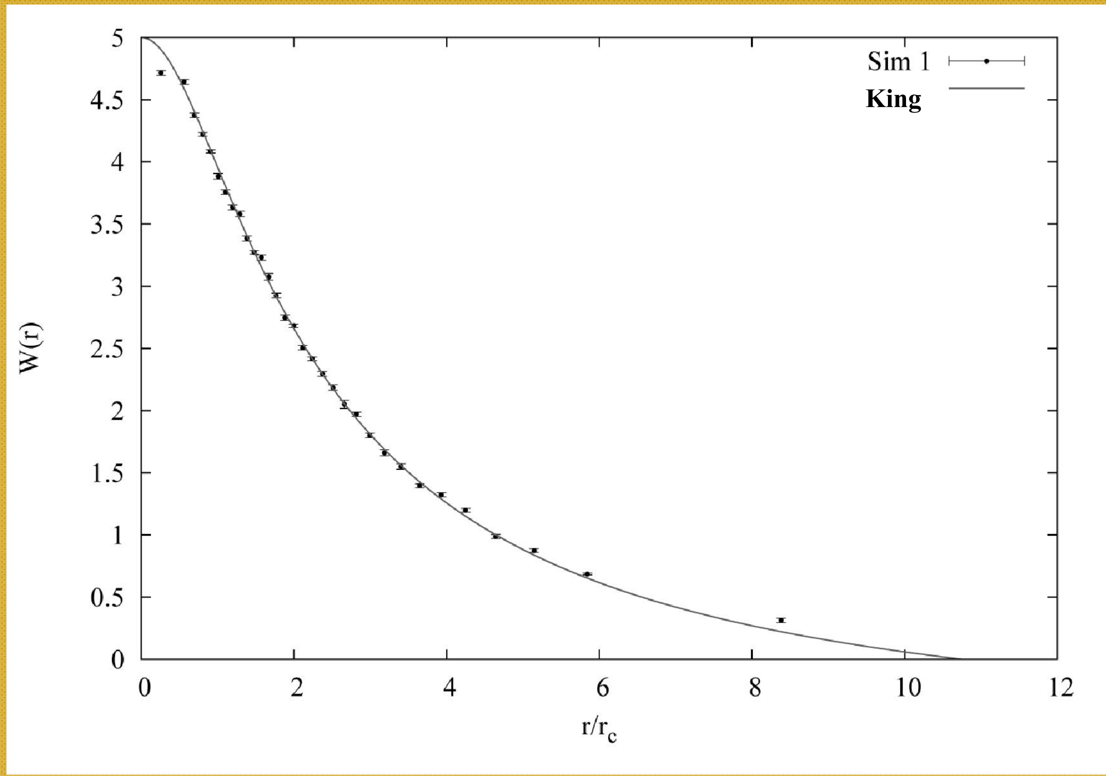


x

r=0.75R ; W=1.663



x



$x=v^2/2\sigma^2$

W_0 =5.033 ; r_c=1.95 pc ;

N=32768 ; m = 1 $M_\odot$.

In each figure, the continuous curve corresponds to the expected behavior

Searching for the effective potential in N-body simulations (models 1 and 2)

1. We consider two models with N=32768 (2^{15}) and N=262144 (2^{18}) stars.
2. We can extract informations valid for the entire cluster (not in a single shell).
3. The distribution function f=exp(-H/kθ) maintains its form all over the cluster.
4. We can check the distribution in order to evaluate the total energy of any single star.

Once obtained this value, we subtract gravitational and kinetic energy and obtain the value of the effective potential.

$$dN(\Sigma, r) = D\, e^{-\Omega(\Sigma)} \sqrt{\Sigma + W(r) - W_\infty}\, d\Sigma\, dV(r); \quad D \text{ is a constant depending on the model}$$

$$\Omega = \Sigma + \Delta; \quad \Delta = -\ln\left(1 - e^{\Sigma - W_\infty}\right) \text{ is the dimensionless effective potential}$$

Extracting the information from the N-body model

$$\sum_{\substack{\text{on all} \\ \text{the shells} \\ r \le r\max}} \frac{\Delta N(s_E, s_r)}{\Delta s_E \sqrt{\langle s_\varepsilon(s_r) \rangle}} = \frac{4}{3} \pi D \beta^3 R_0^3 s_{r\max}^3\, \mathrm{e}^{-\Omega(\Sigma)}$$

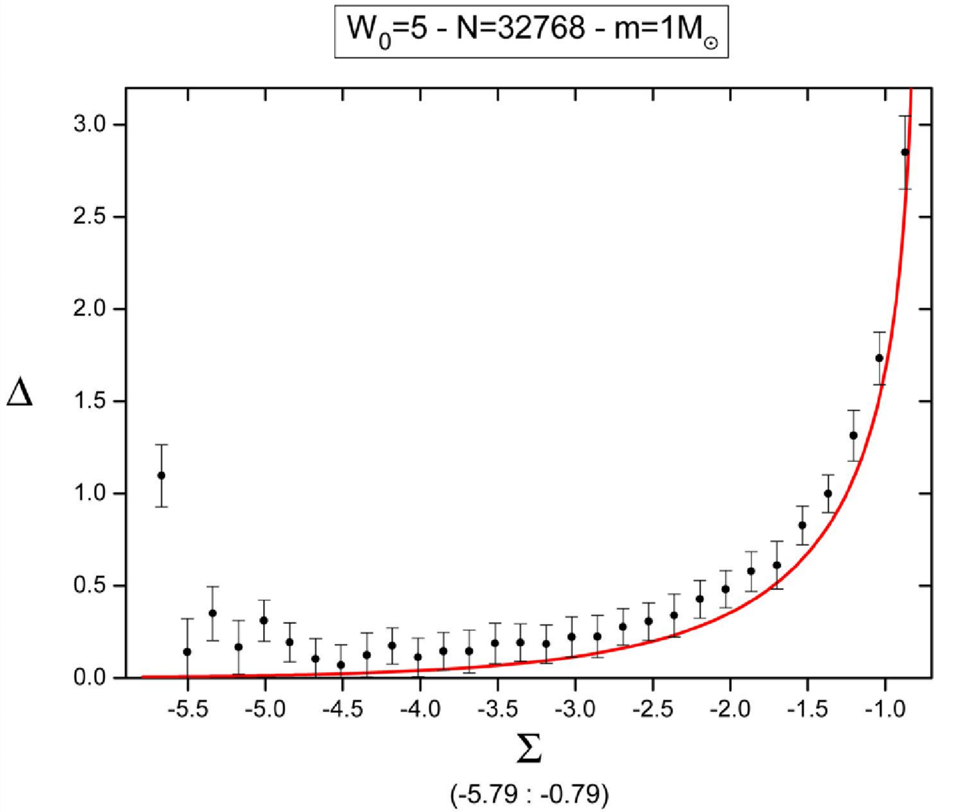
$\beta^2 s_\varepsilon$=star kinetic energy

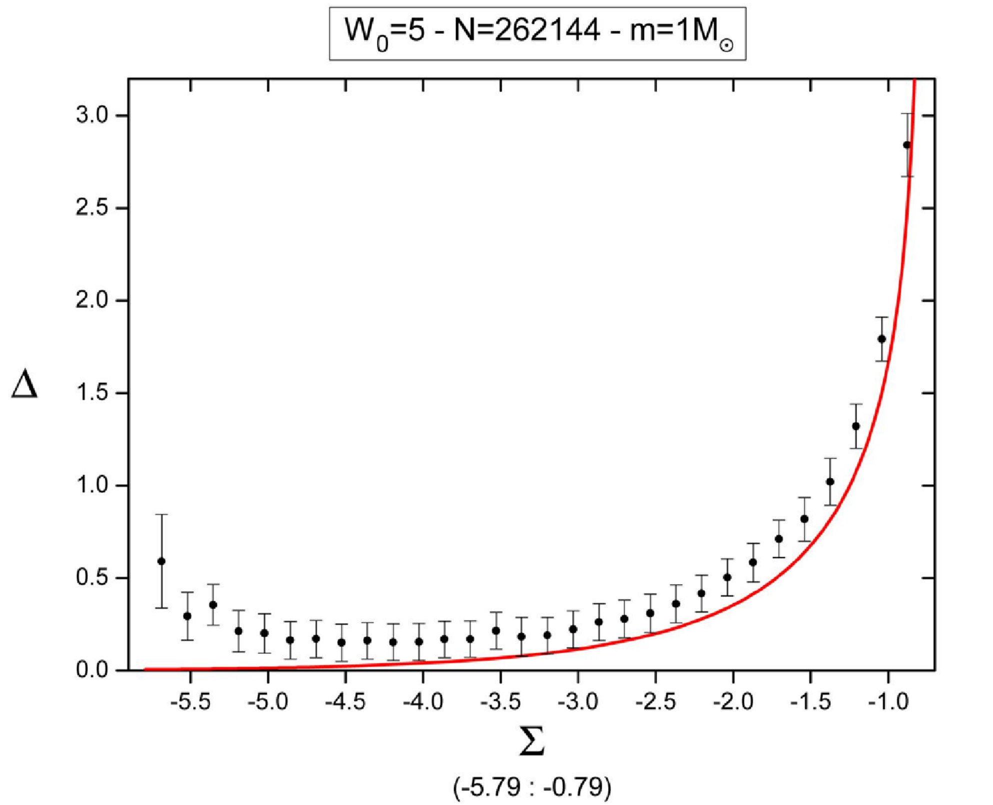
$\Sigma = \beta^2 s_E$; $r = s_r R_0$

R_0=1pc; $\beta = v_0/\sigma$

v_0=1.29km/s

Effective potential





The continuous red curves refer to the expected behavior of the effective potential.

The very small disagreement is due to transition from continuous to discrete system depending by the choice of the radial-energy mesh. This fact becomes critical in the center of the cluster (at low energies).

Conclusions

- Analysis of data from Harris Catalogue ***excludes*** the possibility of any coincidence between the value corresponding to the peak of the W_0-distribution of GCs and the old critical value of the onset of gravothermal instability (W_0=7.4), as requested from the evolution of the GC population.
- The model is selfconsistent and admits ***regions with positive and negative specific heat*** which can exchange energy and produce gravothermal instability without the necessity of an external bath, improving the Lynden-Bell & Wood model.
- The old critical value is modified by the presence of the ***effective potential*** and now coincides with the peak value of the observed distribution, removing the difference outlined by Katz. **This is an osservative evidence of the presence of the effective potential.**

Problems and perspectives

- The W_0-distribution of GSs (from Harris Catalogue) encourages the stability limit W_0=6.91 beyond any doubt. Considering a multi-mass function probably increases the spread between these two values (W_0 corresponding to the peak of the distribution decreases to lesser values).
- New measurements about ***transversal star velocities*** can allow to analyze the real kinematics in globular clusters (at least in a single shell), giving the possibility to check the existence of the effective potential from observational data, instead of using numerical simulations.
- It is necessary to increase/improve observational data in different systems in order to obtain better W_0-distributions of GCs in M31 and NGC5128, able to detect fainter objects and avoiding possible observative selection effects. **This could demonstrate the existence of a unique W_0-distribution of GCs valid for every galaxies**.